# REPRESENTATIONS OF THE GENERAL SYMMETRIC GROUP AS LINEAR GROUPS IN FINITE AND INFINITE FIELDS*

BY

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1. In a series of articles in the Berliner Berichte, beginning in 1896, FROBENIUS has developed an elaborate theory of group-characters and applied it to the representation of a given finite group $G$ as a non-modular linear group. Later, BURNSIDE† approached the subject from the standpoint of continuous groups. The writer has shown‡ that the method employed by BURNSIDE may be replaced by one involving only purely rational processes and hence leading to results valid for a general field. The last treatment, however, expressly excludes the case in which the field has a modulus which divides the order of $G$. The exclusion of this case is not merely a matter of convenience, nor merely a limitation due to the particular method of treatment; indeed,§ the properties of the group-determinant differ essentially from those holding when the modulus does not divide the order of $G$.‖ Thus when $G$ is of order $q!$, the general theory gives no information as to the representations in a field having a modulus $\leqq q$, whereas the case of a small modulus is the most important one for the applications.

The present paper investigates the linear homogeneous groups on $m$ variables, with coefficients in a field $F$, which are simply isomorphic with the symmetric group on $q$ letters. The treatment is elementary and entirely independent of the papers cited above; in particular, the investigation is made for all moduli without exception. The principal result is the determination of the minimum value of the number of variables. It is shown that $m \geqq q-1$ or $m \geqq q-2$, according as $F$ has not or has a modulus $p$ which divides $q$ (§§ 8–21). There

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† Proceedings of the London Mathematical Society, vol. 29 (1898), pp. 207–224, 546–565; vol. 35 (1902), pp. 206–220.

‡ These Transactions, vol. 3 (1902), pp. 285–301.

§ DICKSON, Proceedings of the London Mathematical Society, vol. 35 (1902), p. 68.

‖ Since the present paper was written, the writer has obtained some general results on the outstanding case in which the modulus divides the order of $G$, these Transactions, vol. 8 (1907), pp. 389–398; Bulletin of the American Mathematical Society, vol. 13 (1907), pp. 477–488.

is a representation on $q-1$ variables, except in the trivial case $q=p=2$ (§ 2); while for any field having a modulus which divides there is a representation on $q-2$ variables, except in the special cases $q \leqq 4$ (§ 4).

The determination of the possible representations is made for a case considerably more general than the special one found to be sufficient in establishing the minimum value of $m$ (cf. § 11). We thereby gain a better insight into the nature of the general problem. The question of the equivalence, under linear transformation, of the resulting linear groups is treated in §§ 22–24.

The most essential point in the proof leading to a minimum for $m$ may be seen in a typical case in § 13. After the preliminary normalizations, the transformations involve two sets of parameters. With these we form two rectangular matrices $A$, $B$. Then the conditions on the parameters are equivalent to the condition that the product $AB$ shall equal a known matrix $M$. The representation on $m$ variables is thus possible if and only if the elements of the two matrices $A$ and $B$ can be determined in the field so that $AB=M$. The problem is thus reduced to a simple question of determinants.

2. The symmetric group $S_q$ on $q$ letters is isomorphic with the group of the transformations

$$x_i' = x_{\alpha_i} \qquad (\alpha\text{'s a permutation of } 1, \cdots, q). \tag{1}$$

In place of $x_q$ we introduce the new variable

$$\omega = x_1 + \cdots + x_q, \tag{2}$$

and then suppress $\omega$. According as $\alpha_q = q$ or $\alpha_t = q\,(t<q)$, we get

$$x_i' = x_{\alpha_i} \qquad (\alpha\text{'s a permutation of } 1, \cdots, q-1), \tag{3}$$

$$x_i' = -\sum_{j=1}^{q-1} x_j, \quad x_i' = x_{\alpha_i}\ [\text{the } \alpha_i\ (i \neq t) \text{ a permutation of } 1, \cdots, q-1 \text{ with } \alpha_t \text{ omitted}]. \tag{4}$$

Except when $q=2$ and the coefficients are taken modulo 2, (4) is not the identity $I$, and the isomorphism is simple. Henceforth we shall take $q>2$.

3. The transformations (1) leave invariant (2) and

$$\omega_2 = \sum x_i x_j, \qquad \omega_3 = \sum x_i x_j x_k \quad (i, j, k = 1, \cdots, q;\ i<j<k\cdots). \tag{5}$$

Eliminating $x_q$ from $\omega_2$ by means of (2), we get

$$\omega_2 = \omega \sum_{i=1}^{q-1} x_i - \phi_{q-1},$$

where

$$\phi_{q-1} = \sum x_i^2 + \sum x_i x_j \qquad (i, j = 1, \cdots, q-1;\ i<j). \tag{6}$$

Hence $\phi$ is an invariant (in fact, the only quadratic invariant) of the group of

the transformations (3), (4). For the new variables

$$y_{q-1} = x_{q-1}, \qquad y_i = x_i - x_{q-1} \qquad (i = 1, \cdots, q-2), \tag{7}$$

we have

$$\phi_{q-1}(x) = \phi_{q-2}(y) + q y_{q-1} \sum_{i=1}^{q-2} y_i + \tfrac{1}{2} q(q-1) y_{q-1}^2. \tag{8}$$

We assume that the coefficients of our transformations and invariants may be reduced modulo $p$, where $p$ is any fixed divisor of $q$. If $p$ is odd, or if both $p$ and $q/p$ are even, the second member of (8) reduces to $\phi_{q-2}(y)$. The possibility of this reduction of the number of variables follows directly from the fact that the discriminant of (6) equals $q$.

4. We next discuss the nature of transformations (3), (4), when expressed on the variables (7). We show that if the coefficients be reduced modulo $p$, a divisor of $q$, then $y'_k (k < q-1)$ is a linear function of $y_1, \cdots, y_{q-2}$. Now $y'_k = x'_k - x'_{q-1}$. For (3), this equals $x_{\alpha_k} - x_{\alpha_{q-1}}$ and hence, by (7), equals a function of the $y_i (i < q-1)$. A like result holds for (4) if $k$ and $q-1$ are both distinct from $t$. If $k = t$ or if $q-1 = t$, $y'_k$ involves $q$ $x$'s with like sign and hence equals a function of the $y$'s in which the coefficient of $y_{q-1}$ is $\pm q$. Hence $S_{q!}$ is simply or multiply isomorphic with a linear homogeneous group $L_{q-2,\,p}$ on $q-2$ variables modulo $p$. The substitutions of $S$ which correspond to the identity in $L$ form an invariant subgroup $J$ of $S$. For $q > 4$, $J$ is the identity or else it contains all even substitutions on the $q$ letters. But for $q > 4$, $[x_1 x_2 x_3] \sim [y_1 y_2 y_3]$, so that the isomorphism is simple. For $q = 3$,

$$[x_1 x_2 x_3] \sim I, \qquad [x_1 x_2] \sim y'_1 = -y_1,$$

so that $L_{1,3}$ is of order 2, and the isomorphism is $(3, 1)$. For $q = 4$,

$$[x_1 x_3][x_2 x_4] \sim y'_1 = -y_1, \qquad y'_2 = -2y_1 - y_2;$$
$$[x_1 x_4][x_2 x_3] \sim y'_1 = -y_1 - 2y_2, \qquad y'_2 = -y_2;$$
$$[x_1 x_2] \sim [y_1 y_2], \qquad [x_1 x_3] \sim y'_1 = -y_1, \qquad y'_2 = -y_1 + y_2.$$

Hence $S_{4!}$ is simply isomorphic with $L_{2,4}$, but has $(4, 1)$ isomorphism with $L_{2,2}$. Moreover, $S_{4!}$ cannot be represented as a binary group in a finite or infinite field having modulus 2. Indeed, the transformations of period 2 are conjugate with $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$, which is commutative only with $\left(\begin{smallmatrix} a & 0 \\ c & a \end{smallmatrix}\right)$. The square of the latter is $\left(\begin{smallmatrix} a^2 & 0 \\ 0 & a^2 \end{smallmatrix}\right)$. But $a^4 = 1$ requires $a = 1$ in any field with modulus 2. Hence there is no binary transformation of period 4 commutative with $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$, whereas $S_{4!}$ has a cyclic subgroup of order 4.

THEOREM. *For $q > 4$, the symmetric group on $q$ letters is simply isomorphic with a linear homogeneous group on $q-2$ variables with integral coefficients taken modulo $p$, where $p$ is any integral divisor $> 1$ of $q$. For*

*$q=4$, the result holds if $p=4$, but not if $p=2$. The case $q=3$ is exceptional since a unary homogeneous group is commutative.*

5.* If $p$ is an odd prime dividing $q$, the group $L_{q-2,p}$ has (§ 3) the invariant $\phi_{q-2}(y)$ of discriminant $q-1\equiv -1 \pmod p$. Hence the invariant can be reduced by a linear transformation with integral coefficients to

$$(9)\qquad Y_\epsilon \equiv y_1^2+\cdots+y_{q-3}^2+\epsilon y_{q-2}^2, \qquad \left(\frac{\epsilon}{p}\right)=\left(\frac{-1}{p}\right)\left(\frac{2}{p}\right)^{q-2}.$$

According as $\epsilon$ is a square or a not-square in a field $F$, the group defined by the invariant $Y_\epsilon$ is called a first or a second orthogonal group in $F$. In particular, *for the $GF[p^n]$ the group is a first or second orthogonal group according as*

$$(10)\qquad \left(\frac{-1}{p}\right)^n\left(\frac{2}{p}\right)^{n(q-2)}=+1 \text{ or } -1.$$

Next let $p=2$. The quadratic form

$$(11)\qquad Q_{2m}=\sum\xi_i^2+\sum\xi_i\xi_j \qquad (i,j=1,\cdots,2m;\ i<j)$$

can be reduced by a linear transformation with integral coefficients to

$$(12)\qquad f_c=\xi_1\xi_2+\cdots+\xi_{2m-1}\xi_{2m}+c(\xi_1^2+\xi_2^2),$$

where $c\equiv 0$ or $1 \pmod 2$ according as 2 is a quadratic residue or non-residue of the discriminant $2m+1$ of $Q_{2m}$. For the $GF[2^n]$, $f_1$ can be reduced to $f_0$ if and only if $n$ be even. The group defined by the invariant $f_0$ is called the first hypoabelian group; that defined by $f_1$ with $\xi_1\xi_2+\xi_1^2+\xi_2^2$ irreducible is called the second hypoabelian group. Hence *the group defined by the invariant $Q_{2m}$ in the $GF[2^n]$ can be transformed into the first or second hypoabelian group according as*

$$(13)\qquad \left(\frac{2}{2m+1}\right)^n=+1 \text{ or } -1.$$

Next, $\xi_0^2+Q_{2m}$ can be reduced to $\xi_0^2+f_0$, since the operation

$$\xi_0'=\xi_0+\xi_1+\xi_2, \qquad \xi_i'=\xi_i \qquad (i>0)$$

replaces $\xi_0^2+f_c$ by $\xi_0^2+f_{c+1}$. Now for $p=2$, $q/2$ odd, the analysis in § 3 failed to exhibit a quadratic invariant for the group on $y_1,\cdots,y_{q-2}$, but led to the invariant $\phi_{q-2}(y)+y_{q-1}^2$ for the group on $y_1,\cdots,y_{q-1}$. Hence the latter group can be transformed into one with the invariant

$$y_1y_2+y_3y_4+\cdots+y_{q-3}y_{q-2}+y_{q-1}^2.$$

* The results of §§ 5, 6, are not employed in the subsequent sections.

By an elementary discussion (the writer's *Linear Groups*, p. 200), the group is seen to be simply isomorphic with an abelian linear group on $q-2$ variables. An independent proof follows from § 6.

THEOREM. *There exist representations of the symmetric group on q letters as linear homogeneous groups as follows:*

(i) *for q odd, a first or second hypoabelian group in the* $GF[2^n]$ *on* $q-1$ *variables, according as* $(2/q)^n = +1$ *or* $-1$;

(ii) *for* $q = 4l > 4$, *a first or second hypoabelian group in the* $GF[2^n]$ *on* $q-2$ *variables, according as* $[2/(q-1)]^n = +1$ *or* $-1$;

(iii) *for* $q = 4l+2 > 2$, *an abelian linear group in any field* $F_2$ *having modulus* 2 *on* $q-2$ *variables.*

By considering the monodromy group of the equation for the bisection of the periods of certain abelian functions, JORDAN has shown (*Traité des Substitutions*, p. 364, § 498) that the symmetric group on $q$ letters is simply isomorphic with an abelian linear group modulo 2 on $2k$ variables, $k$ being the greatest integer in $\frac{1}{2}(q-1)$. The exceptional character of the case $q=4$ was overlooked. The theorem obtained above shows that if $q \neq 4l+2$ the abelian group may be taken to be a hypoabelian group.

Additional properties of our linear groups may be obtained by employing the further invariants * (5). Proceeding with $\omega_3$ and $\omega_4$ as we did with $\omega_2$ in § 3, we obtain functions independent of $y_{q-1}$ if and only if $p=2$ and $p=3$, respectively. Thus for $p=2, 3$, the group $L_{q-2,p}$ has the respective invariants

$$\sum y_j^2 y_k, \quad \sum y_j^2 y_k y_l \quad (j,k,l=1,\cdots,q-2;\ j,k,l \text{ distinct};\ k<l). \tag{14}$$

6. The fact that the symmetric group on $2k$ letters ($2k>4$) is simply isomorphic with an abelian group modulo 2 on the variables $\xi_i, \eta_i$ ($i=1,\cdots,k-1$) may also be shown by the following correspondence of generators:

$$[12] \sim L_1', \ [23] \sim L_1, \ [34] \sim V_{12}, \ [45] \sim L_2, \ [56] \sim V_{23}, \ \cdots,$$
$$[2i, 2i+1] \sim L_i, \ [2i+1, 2i+2] \sim V_{ii+1}, \ \cdots,$$
$$[2k-2, 2k-1] \sim L_{k-1}, \ [2k-1, 2k] \sim L_{k-1}',$$

in which the $L$'s and $V$'s alternate. Here

$$L_i\colon \ \xi_i' = \xi_i + \eta_i; \qquad L_i'\colon \ \eta_i' = \eta_i + \xi_i;$$
$$V_{ij} \equiv V_{ji}\colon \ \eta_i' = \eta_i + \xi_i + \xi_j, \ \eta_j' = \eta_j + \xi_i + \xi_j.$$

Since each transformation is of period 2, and $V_{ij}$ is commutative with $L_i'$, $L_j'$, $V_{jk}$, while $L_i' L_i$ and $L_i V_{ij}$ are of period 3, the above transformations satisfy

* If we assume that the variables, as well as the coefficients, are integers taken modulo 2, then the group $L_{q-2,2}$ has, for $q = 8t+2$, the invariant

$$\sum y_j y_k y_l + \sum y_i y_j y_k y_l \qquad (i,j,k,l=1,\ldots,q-2;\ i<j<k<l)$$

MOORE's set* of generational relations for the abstract form of the symmetric group. If we transform by $L_1 L_2 \cdots L_{k-1}$, we obtain the generators

$$M_1,\ L_1,\ W_{12},\ L_2,\ W_{23},\ L_3,\ \cdots,\ L_{k-1},\ M_{k-1},$$

and hence have a direct generalization of the result for $k = 6$ in *Linear Groups*, §118, page 99. Here $W_{12} = S$.

For $k = 4$, the group generated by $L_1'$, $L_1$, $\cdots$ has the (single) quadratic invariant†

$$\xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3 + \xi_1 + \xi_2 + \xi_3 + \eta_1 + \eta_3.$$

The abelian transformation $L_2' L_1' L_3 Q_{13}$ replaces this by

$$\xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3.$$

Hence (in accord with § 5) $S_{8!}$ is representable as a first hypoabelian group. But the total first hypoabelian group modulo 2 on 6 variables is of order 8!.

THEOREM.‡ *The symmetric group on* 8 *letters is simply isomorphic with the total first hypoabelian group modulo* 2 *on* 6 *variables.*

7. LEMMA. *If the direct product of two groups A and B contains a subgroup G simply isomorphic with the symmetric group on n letters, then A or B contains a subgroup of that type.*

Let the operations of $G$ be $a_1 b_1$, $a_2 b_2$, $\cdots$. Then the distinct $a$'s form a group $\alpha$, the distinct $b$'s a group $\beta$. The largest group $\alpha'$ common to $\alpha$ and $G$ is evidently invariant in each; the largest group $\beta'$ common to $\beta$ and $G$ is invariant in each. As well known, the quotient-groups

$$\frac{G}{\{\alpha', \beta'\}},\qquad \frac{\alpha}{\alpha'},\qquad \frac{\beta}{\beta'}$$

are simply isomorphic. By hypothesis, $G$ is simply isomorphic with $S_\nu$, $\nu = n!$. Hence the only invariant subgroups of $G_\nu$ are itself, the identity $I$, the group $G_{\frac{1}{2}\nu}$ of the alternating type, and for $n = 4$ a "four group" $G_4$. The lemma is obvious if either $\alpha'$ or $\beta'$ is $G$ itself, or if each is $I$, or if each is $G_{\frac{1}{2}\nu}$ (whence $n = 2$). Next let $\alpha' = G_{\frac{1}{2}\nu}$, $\beta' = I$. Then $G$ is composed of the operations of $\alpha' = G_{\frac{1}{2}\nu}$ and the products of $\frac{1}{2}\nu$ further $a$'s by an operation $b$ of period 2, so that $G$ is simply isomorphic with $\alpha$. Finally, if $n = 4$, and $\alpha' = G_4$, then $\beta'$ cannot be of type $G_4$ or $G_{\frac{1}{2}\nu}$, so that $\beta' = I$; hence $\alpha$ is of order 4! and is simply isomorphic with $G$.

*Proceedings of the London Mathematical Society, vol. 28 (1896), p. 357. For $k > 2$ the isomorphism is simple in view of his theorem $C$.

† For $k = 5$, this abelian group on 8 variables has no linear or quadratic invariant. The subgroup corresponding to the alternating group on $1, \cdots, 10$ has no quadratic invariant.

‡ For a proof based on other principles, see the writer's note in the Bulletin of the American Mathematical Society, vol. 13, (1907), pp. 386-389.

8. THEOREM. *For $q$ odd, the symmetric group on $q$ letters cannot be represented as a linear homogeneous group on $q-2$ variables with coefficients in a finite or infinite field $F_2$ having modulus 2.*

It follows from the theory of canonical forms that every transformation of period 2 of the general linear homogeneous group $H$ on $q-2$ variables in $F_2$ is conjugate within $H$ with one of the transformations

$$(15) \qquad \xi_i' = \xi_i, \quad \xi_{r+i}' = \xi_{r+i} + \xi_i, \quad \xi_j' = \xi_j \qquad (i = 1, \cdots, r; j = 2r+1, \cdots, q-2),$$

where $r$ is an integer such that $2r \leqq q-2$, whence $r \leqq \frac{1}{2}(q-3)$. The matrix of (15) and the matrix of the general transformation of $H$ commutative with (15) may be given the compact notations

$$(16) \qquad \begin{matrix} I & O & O \\ I & I & O, \\ O & O & I \end{matrix} \qquad \begin{matrix} A & O & O \\ C & A & D, \\ E & O & B \end{matrix}$$

in which each $I$ is a unit or identity matrix, each $O$ a matrix all of whose elements are zero, while $A$ and $B$ are square matrices of orders $r$ and $q-2-2r$, respectively.

Suppose now that $H$ has a subgroup $G_{q!}$ of symmetric type. We may assume that the latter contains a transformation (15) corresponding to the transposition $[l_1 l_2]$, and that the group $K$ of the transformations $(16_2)$ contains a subgroup $G_\omega$, $\omega = (q-2)!$, simply isomorphic with the symmetric group on $l_3, \cdots, l_q$. The group $K$ has the invariant subgroup $J$,

$$J: \begin{matrix} I & O & O \\ C & I & D, \\ E & O & I \end{matrix} \qquad Q: \begin{matrix} A & O & O \\ O & A & O, \\ O & O & B \end{matrix}$$

the quotient group $K/J$ being simply isomorphic with $Q$. The largest group common to $J$ and $G_\omega$ is invariant in the latter. Now every transformation of $J$ is of order a power of 2. But, for $n \neq 4$, $S_{n!}$ contains no invariant subgroup of order a power of 2. Since $q$ is odd, it follows that $G_\omega$ is simply isomorphic with a subgroup of $Q$, and hence (§ 7) with a subgroup of one of the linear groups $A$ and $B$ on $r$ and $q-2-2r$ variables, respectively. Since $1 \leqq r \leqq \frac{1}{2}(q-3)$, we have $q-2-2r \leqq q-4$, and, if $q \geqq 5$, $r \leqq q-4$. Hence the theorem follows by induction from $q-2$ to $q$, it being obviously true for $q = 3$, since a unary homogeneous group is commutative.

For $q$ even, we apply the preceding theorem to the subgroup leaving $l_q$ fixed and obtain the

COROLLARY. *For $q$ even, $S_{q!}$ cannot be represented as a linear homogeneous group in $F_2$ on $q-3$ variables.*

The preceding theorem and corollary form a sequel to the results of §§ 2, 4 for the case when the modulus is 2.

9. Henceforth we discuss the representations of $S_{q'}$ as an $m$-ary linear homogeneous group in a field $F$ not having modulus 2. Within the general linear homogeneous group $GLH(m, F)$, any transformation of period 2 is conjugate with one of the form $\Pi C_i$, where $C_i$ alters only $\xi_i$, whose sign it changes. We thus assume that the transposition $[12]$ corresponds to $C_1 C_2 \cdots C_r$. Then each transposition corresponds to a transformation $(\alpha_{ij})$ with the characteristic determinant

$$|\alpha - \rho I| = (-1-\rho)^r (1-\rho)^{m-r}. \tag{17}$$

To $[34]$ corresponds a transformation $S$ commutative with $C_1 \cdots C_r$. Transforming $S = S_{1,\ldots,r} S'_{r+1,\ldots,m}$ by a transformation of type $T_{1,\ldots,r} T'_{r+1,\ldots,m}$, we obtain a product of $r$ of the $C_i$. Hence*

$$[12] \sim C_1 \cdots C_r, \qquad [34] \sim C_1 \cdots C_s C_{r+1} \cdots C_{2r-s} \quad (0 \leqq s < r). \tag{18}$$

Since $[13][24]$ is commutative with $[12][34]$, we get

$$[13][24] \sim R S_{1,\ldots,s,\,2r-s+1,\ldots,m}, \qquad R = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where $\alpha, \cdots, \delta$ are square matrices of order $r-s$, $\alpha$ affecting the variables $\xi_{s+1}, \cdots, \xi_r$, and $\delta$ the variables $\xi_{r+1}, \cdots, \xi_{2r-s}$. We may obtain $[14][23]$ by transforming $[13][24]$ by $[12]$ or as the product $[13][24] \cdot [12][34]$. Identifying the two results, we find that $\alpha = \delta = 0$, and that $S$ is commutative with $C_1 \cdots C_s$. Also $\beta\gamma = \gamma\beta = I$, since $R^2 = I$. Hence

$$[13][24] \sim \begin{pmatrix} O & \beta \\ \beta^{-1} & O \end{pmatrix} \cdot S_{1,\ldots,s} S'_{2r-s+1,\ldots,m}.$$

We may take $\beta = I$, $S = C_1 \cdots C_s$, $S' = C_{2r-s+1} \cdots$, by transforming by

$$(\beta)_{r+1,\ldots,2r-s} T_{1,\ldots,s} T'_{2r-s+1,\ldots,m},$$

where $T$ and $T'$ are suitably chosen. Then (18) are unaltered, and

$$[13][24] \sim C_1 \cdots C_s \prod_{i=1}^{r-s} (\xi_{s+i} \xi_{r+i}) \prod_{i=1}^{r-s-\sigma} C_{2r-s+i} \quad (0 \leqq \sigma \leqq s,\ \sigma \leqq r-s). \tag{19}$$

The number of factors in the final product must be $r-s-\sigma$ since the number of roots $-1$ of the characteristic equation is $2r-2s$, $[13][24]$ being conjugate with $[12][34]$. By (18) and (19), we have

$$[14][23] \sim C_1 \cdots C_s \prod_{i=1}^{r-s} (\xi_{s+i} \xi_{r+i}) \prod_{i=s+1}^{3r-2s-\sigma} C_i. \tag{20}$$

* The terms $C_1 \cdots C_s$ are to be suppressed in the case $s = 0$. Similarly below. For the case $q = 3$, here excluded, we may proceed as in § 16, with $k = 0$.

In applying (18) and (19) simultaneously, it is convenient to have the variables separated into the sets $\xi_{i_1}, \cdots, \xi_{i_6}$, where

$$i_1 = 1, \cdots, \sigma;\ i_2 = \sigma + 1, \cdots, s;\ i_3 = s + 1, \cdots, r;$$

$$(21) \qquad i_4 = r + 1, \cdots, 2r - s;\ i_5 = 2r - s + 1, \cdots, 3r - 2s - \sigma;$$

$$i_6 = 3r - 2s - \sigma + 1, \cdots, m;$$

the set $\xi_{i_1}$ not occurring if $\sigma = 0$, etc. Then (20) may be written

$$(20') \qquad [14][23] \sim (-I)_{i_1}(I)_{i_2}\begin{pmatrix} O & -I \\ -I & O \end{pmatrix}_{i_3, i_4}(-I)_{i_5}(I)_{i_6}.$$

Let $[23] \sim K = (\kappa_{ab})$, $a, b = 1, \cdots, 6$. Since $K$ is commutative with (20'),

$$K = \begin{bmatrix} \kappa_{11} & O & \kappa_{13} & \kappa_{13} & \kappa_{15} & O \\ O & \kappa_{22} & \kappa_{23} & -\kappa_{23} & O & \kappa_{26} \\ \kappa_{31} & \kappa_{32} & \kappa_{33} & \kappa_{34} & \kappa_{35} & \kappa_{36} \\ \kappa_{31} & -\kappa_{32} & \kappa_{34} & \kappa_{33} & \kappa_{35} & -\kappa_{36} \\ \kappa_{51} & O & \kappa_{53} & \kappa_{53} & \kappa_{55} & O \\ O & \kappa_{62} & \kappa_{63} & -\kappa_{63} & O & \kappa_{66} \end{bmatrix}.$$

Now [14] may be obtained by transforming [23] by [13][24] and also by multiplying [23] by [14][23]. Equating the results for the corresponding transformations, we get

$$\kappa_{11} = \kappa_{15} = \kappa_{23} = \kappa_{32} = \kappa_{36} = \kappa_{51} = \kappa_{55} = \kappa_{63} = 0, \qquad \kappa_{34} = -\kappa_{33}.$$

Hence, setting $\kappa = \kappa_{33}$, we have

$$(22) \quad K = UV, \quad U = \begin{bmatrix} O & \kappa_{13} & \kappa_{13} & O \\ \kappa_{31} & \kappa & -\kappa & \kappa_{35} \\ \kappa_{31} & -\kappa & \kappa & \kappa_{35} \\ O & \kappa_{53} & \kappa_{53} & O \end{bmatrix}_{i_1, i_2, i_4, i_5}, \quad V = \begin{pmatrix} \kappa_{22} & \kappa_{26} \\ \kappa_{62} & \kappa_{66} \end{pmatrix}_{i_2, i_6}.$$

We shall employ the symbols

$$(23) \qquad \mu \equiv \left.\begin{matrix} \kappa_{13} \\ \kappa_{53} \end{matrix}\right\}, \qquad \nu \equiv \overbrace{\kappa_{31}\,\kappa_{35}}$$

to denote the square matrices of order $r - s$ defined as follows: The first $\sigma$ rows of $\mu$ are composed of the elements of $\kappa_{13}$, the remaining $r - s - \sigma$ rows are composed of the elements of $\kappa_{53}$; while $\nu$ is formed by annexing columns. By

combining the sets of variables $\xi_{i_1}$ and $\xi_{i_5}$ into one set $\xi_{i_1, 5}$, we have

$$U = \begin{pmatrix} \kappa & -\kappa & \nu \\ -\kappa & \kappa & \nu \\ \mu & \mu & O \end{pmatrix}_{i_3, i_4, i_{1,5}} . \tag{24}$$

Hence $U^2 = I$ if and only if

$$\mu\nu = \tfrac{1}{2}I, \qquad \kappa^2 = \tfrac{1}{4}I. \tag{25}$$

Now [12] [23] is of period 3. Hence must

$$(C_{\sigma+1} \cdots C_s V)^3 = I \tag{26}$$

and $(CU)^3 = I$, where $C = C_1 \cdots C_\sigma C_{s+1} \cdots C_r$. For $U$ given by (22), the conditions for $(CU)^2 = UC$ are seen to be

$$2\kappa_{13}\kappa = -\kappa_{13},\ 2\kappa_{53}\kappa = \kappa_{53},\ 2\kappa\kappa_{31} = -\kappa_{31},\ 2\kappa\kappa_{35} = \kappa_{35},\ \kappa_{35}\kappa_{53} - \kappa_{31}\kappa_{13} = \kappa. \tag{27}$$

The first two of these conditions give

$$\mu\kappa = \delta\mu, \tag{28}$$

where $\delta$ is a square matrix all of whose non-diagonal elements are zero, while in the main diagonal the first $\sigma$ elements are $-\frac{1}{2}$, and the remaining $r - s - \sigma$ elements are $+\frac{1}{2}$.

We next normalize the linear group by a transformation of variables. We multiply all matrices on the left by $M$ and on the right by $M^{-1}$, where $M = (\mu)_{i_3}(\mu)_{i_4}$. Evidently (18)–(20) are unaltered. Then (24) becomes a matrix in which $\kappa$, $\mu$, $\nu$ are replaced by $\mu\kappa\mu^{-1}$, $I$, $\mu\nu$, respectively. Hence by ($25_1$) and (28), the new $U$ is

$$U = \begin{pmatrix} \delta & -\delta & \frac{1}{2}I \\ -\delta & \delta & \frac{1}{2}I \\ I & I & O \end{pmatrix}_{i_3, i_4, i_{1,5}} . \tag{29}$$

The remaining conditions ($25_2$) and (27) are seen to be satisfied. The characteristic equation of (29) is seen to have $-1$ as a root of multiplicity $r - s + \sigma$. Hence by (17) that of $V$ has $-1$ as root of multiplicity $s - \sigma$.

THEOREM. *In every representation of the symmetric group $S_{4!}$ as a linear homogeneous group in a field not having modulus 2, we may, after a suitable transformation of variables, take the correspondence of generators to be* (18)–(20) *and* $[23] \sim UV$, *where $U$ is given by* (29) *and $V$ is a transformation $V_{i_{2,6}}$ subject to condition* (26) *and having $-1$ as a root of multiplicity $s - \sigma$ of its characteristic equation.*

10. A transformation is commutative with (18), (19), and $UV$ if and only

if it has the form

$$T_{i_3}T_{i_4}T_{i_{1,5}}Q_{i_2}R_{i_6}, \qquad T=\begin{pmatrix} A & O \\ O & B \end{pmatrix}, \tag{30}$$

where $A$ and $B$ are square matrices of orders $\sigma$ and $r-s-\sigma$, respectively.

If $q \geqq 6$, let $[56] \sim S$. By transforming the linear group by a transformation of type (30), we may express $S$ as a product of $C$'s, without disturbing the earlier correspondences. Hence we may set

$$[56] \sim \prod_{i=1}^{\sigma_1} C_{s+i}C_{r+i}C_i \cdot \prod_{i=1}^{\sigma_2} C_{s+\sigma+i}C_{r+\sigma+i}C_{2r-s+i} \cdot \prod_{i=1}^{\sigma_3} C_{\sigma+i} \cdot \prod_{i=1}^{\sigma_4} C_{3r-2s+\sigma+i}, \tag{31}$$

where

$$0 \leqq \sigma_1 \leqq \sigma, \qquad 0 \leqq \sigma_2 \leqq r-s-\sigma, \qquad 0 \leqq \sigma_3 \leqq s-\sigma, \qquad 0 \leqq \sigma_4, \tag{32}$$

$$3(\sigma_1+\sigma_2)+\sigma_3+\sigma_4 = r. \tag{33}$$

Now $(18_1)$ and (31) have $2\sigma_1+\sigma_2+\sigma_3$ factors $C$ in common, while the products (18) have $s$ factors in common. Hence

$$2\sigma_1+\sigma_2+\sigma_3 = s. \tag{34}$$

11. The main object of the paper is to determine the minimum number of variables upon which the symmetric group $S_{q!}$ can be represented. For this purpose it will be seen (§§ 19, 20) to be sufficient to know the representations in the case $r=1$, whence $s=0$. We shall assume merely that $s=0$, as the specialization of $r$ does not materially simplify the discussion.

For $s=0$, we have $\sigma=0$. Then $V=I$ since $V^2=I$ and since all the roots of the characteristic equation are $+1$. Hence

$$[12] \sim C_1 \cdots C_r, \qquad [34] \sim C_{r+1} \cdots C_{2r}, \qquad [23] \sim U_{1,\ldots,3r}, \tag{35}$$

the matrix of $U$ being given by (29) for $\delta = \frac{1}{2}I$. Hence

$$U_{1,\ldots,3r}\colon\ \xi_i' = \xi_i + \omega_i,\ \ \xi_{r+i}' = \xi_{r+i} + \omega_i,\ \ \xi_{2r+i}' = \xi_{2r+i} - 2\omega_i \quad (i=1,\cdots,r), \tag{36}$$

$$\omega_i = -\tfrac{1}{2}\xi_i - \tfrac{1}{2}\xi_{r+i} + \tfrac{1}{2}\xi_{2r+i}, \qquad \omega_i' = -\omega_i \text{ under } U. \tag{37}$$

In § 10 we now have $\sigma_1=\sigma_2=\sigma_3=0$, $\sigma_4=r$. Hence if $q \geqq 6$,

$$[56] \sim C_{3r+1} \cdots C_{4r}. \tag{38}$$

Since $[36][45]$ is commutative with $[12]$ and $[34][56]$, and since its transform by $[56]$ and its product by $[34][56]$ both equal $[35][46]$, we have

$$[36][45] \sim T = A_{1,\ldots,r}B_{r+1,\ldots,2r,\,3r+1,\ldots,4r}C, \qquad B=\begin{pmatrix} O & \beta \\ \beta^{-1} & O \end{pmatrix},$$

where $\beta$ is a square matrix of order $r$, and

$$C=(\gamma_{ij}), \qquad C^2=I \quad (i,j=2r+1,\cdots,3r,\ 4r+1,\cdots,m). \tag{39}$$

Transforming the group by $(\beta)_{3r+1,\ldots,4r}$, we may set $\beta = I$. Then the characteristic determinant $B_\rho$ of $B$ equals $(\rho^2-1)^r$. If $A_\rho = 0$ has $t$ roots $-1$, then $C_\rho = 0$ has $r-t$ roots $-1$, since $[36][45]$ is conjugate with $[12][34]$. Then $[12][36][45]$ has $r-t+r+r-t$ roots $-1$, but is conjugate with $[12][34][56]$ with $3r$ roots $-1$; hence $t=0$, $A=I$. Thus

$$(40) \qquad [36][45] \sim T = \prod_{i=1}^{r} (\xi_{r+i}\xi_{3r+i}) \cdot C \quad (C_\rho \text{ with } r \text{ roots } -1).$$

Let $[45] \sim L$. We shall determine $L$ by a device which brings to light its essential properties. The product

$$[36][45] \cdot [23] \cdot [34] \cdot [56]$$

transforms $[23]$ into $[45]$. Let $K = C_{r+1} \cdots C_{2r} C_{3r+1} \cdots C_{4r}$. Then the product $P = TUK$ transforms $U$ into $L$. Hence $P^{-1} = KUT$ replaces the invariants of $U$ by invariants of $L$, and replaces $\omega_i$, given by (37), by a function which $L$ must multiply by $-1$, namely,

$$\tau_i = -\tfrac{1}{4}\xi_i - \tfrac{1}{4}\xi_{r+i} - \tfrac{1}{4}\xi_{2r+i} + \tfrac{1}{2}\xi_{3r+i} + \tfrac{1}{2}\sum_{j=1}^{r} \gamma_{2r+i,\,2r+j}(\xi_j - \xi_{r+j}) + \tfrac{1}{2}\sum_{j=4r+1}^{m} \gamma_{2r+i,\,j}\xi_j.$$

Applying $P^{-1}$ to the invariants $\xi_{r+i} + \frac{1}{2}\omega_i$ and $\xi_{3r+i}$ of $U$, we get

$$-\xi_{3r+i} + \tfrac{1}{2}\tau_i, \ \omega_i \qquad (i=1, \cdots, r).$$

Since $C$ replaces $\sum \gamma_{ij}\xi_j$ by $\xi_i$ by (39), $P^{-1}$ replaces the invariants

$$\sum_{j=1}^{r} \gamma_{i,\,2r+j}(\xi_{2r+j} - \omega_j) + \sum_{j=4r+1}^{m} \gamma_{ij}\xi_j \quad (i=2r+1, \cdots, 3r, 4r+1, \cdots, m)$$

of $U$ by the following invariants of $L$:

$$(41) \quad \xi_i - \xi_{r+i} - \sum_{j=1}^{r} \gamma_{2r+i,\,2r+j}\tau_j \ (i=1, \cdots, r), \quad \xi_i - \sum_{j=1}^{r} \gamma_{i,\,2r+j}\tau_j \ (i=4r+1, \cdots, m).$$

Since $[45]$ is commutative with $[12]$, $L$ must be commutative with $C_1 \cdots C_r$. Hence $L$ replaces $\xi_1, \cdots, \xi_r$ by functions of the same, while the function by which $L$ replaces $\xi_t (t > r)$ is independent of $\xi_1, \cdots, \xi_r$. Hence by the invariance of $\omega_1, \cdots, \omega_r$, $L$ leaves fixed $\xi_1, \cdots, \xi_r$. But $L$ must multiply each $\tau_i$ by $-1$; hence the $\tau_i$ are independent of $\xi_1, \cdots, \xi_r$, so that

$$(42) \qquad \gamma_{2r+i,\,2r+i} = \tfrac{1}{2}, \qquad \gamma_{2r+i,\,2r+j} = 0 \quad (i, j=1, \cdots, r; \ i \neq j).$$

Suitable linear combinations of the above invariants of $L$ give

$$\xi_i, \ \xi_{r+i} + \tfrac{1}{2}\tau_i, \ \xi_{2r+i} - \xi_{r+i}, \ \xi_{3r+i} - \tfrac{1}{2}\tau_i \qquad (i=1, \cdots, r).$$

In view of these and $(41_2)$, the explicit form of $L$ is

$$(43)\quad \delta\xi_{r+i}=\tau_i,\quad \delta\xi_{2r+i}=\tau_i,\quad \delta\xi_{3r+i}=-\tau_i\,(i=1,\cdots,r),\quad \delta\xi_i=-2\sum_{j=1}^{r}\gamma_{i,\,2r+j}\,\tau_j \qquad (i=4r+1,\cdots,m),$$

$$(44)\quad \tau_i=-\tfrac{1}{2}\xi_{r+i}-\tfrac{1}{4}\xi_{2r+i}+\tfrac{1}{2}\xi_{3r+i}+\tfrac{1}{2}\sum_{j=4r+1}^{m}\gamma_{2r+i,\,j}\,\xi_j,$$

where $\delta\xi\equiv\xi'-\xi$. Now $L$ actually multiplies each $\tau_i$ by $-1$ if and only if*

$$(45)\quad \sum_{j=4r+1}^{m}\gamma_{2r+i,\,j}\gamma_{j,\,2r+k}=\tfrac{3}{4}\delta_{ik} \qquad (i,k=1,\cdots,r),$$

where, in KRONECKER's notation, $\delta_{ii}=1$, $\delta_{ik}=0\,(i\neq k)$.

Since $L$ was shown to leave each $\omega_i$ invariant, and since $U$ evidently leaves each $\tau_i$ invariant, $L$ and $U$ are commutative. Since $LC_{r+1}\cdots C_{2r}$ and $LC_{3r+1}\cdots C_{4r}$ replace $\tau_i$ by $\xi_{r+i}$ and $-\xi_{3r+i}$, respectively, they are easily seen to have period 3. Hence the correspondence given by (35), (38), and $[45]\sim L$, define a simple isomorphism between $S_{6!}$ and an $m$-ary linear group.

12. We may now readily treat the case when $q$ is general:

$$(46)\quad q=4\kappa+\kappa' \qquad (0\leqq\kappa'<4).$$

By the proof for [56], any transposition $[ij]$, $i>4$, $j>4$, corresponds to a transformation on the variables $\xi_k$ $(k>3r)$. Hence we may set

$$(47)\quad \begin{gathered}[56]\sim\Pi C_{3r+i},\qquad [67]\sim U_{3r+1,\ldots,6r},\qquad [78]\sim\Pi C_{4r+i};\\ [9,10]\sim\Pi C_{6r+i},\quad [10,11]\sim U_{6r+1,\ldots,9r},\quad [11,12]\sim\Pi C_{7r+i};\ \cdots\\ [4\kappa-3,4\kappa-2]\sim\Pi C_{(3\kappa-3)r+i},\quad [4\kappa-2,4\kappa-1]\sim U_{(3\kappa-3)r+1,\ldots,3\kappa r},\\ [4\kappa-1,4\kappa]\sim\Pi C_{(3\kappa-2)r+i},\end{gathered}$$

where for each product $i=1,\cdots,r$. Each of the $\kappa$ "triples" (35) and (47) defines a simple isomorphism between the symmetric group on four letters and a linear group on $3r$ variables. It remains to insert the connecting "links" $[45]\sim L_1$, $[89]\sim L_2$, $\cdots$, and when $\kappa'>0$ in (46) to extend the series beyond these $\kappa$ triples (§§ 14–16).

In § 11 the only normalization was the transformation by $(\beta)_{3r+1,\ldots,4r}$. We here accomplish this normalization by transforming the linear group by

$$(\beta)_{3r+1,\ldots,4r}(\beta)_{4r+1,\ldots,5r}(\beta)_{5r+1,\ldots,6r},$$

and hence preserve the correspondences (35) and (47). Then $L_1$ is given by (43), subject to (44) and (45). Now $L_1$ must be commutative with all the transformations (47) except the first. Thus $L_1$ does not affect the variables

* These conditions also follow from (39) and (42).

$\xi_i$ $(i = 6r+1, \cdots, 3\kappa r)$, in view of the triples other than the first, nor $\xi_i$ $(i = 4r+1, \cdots, 5r)$, in view of [78]. Finally, $L_1$ must be commutative with $U = U_{3r+1, \cdots, 6r}$. The necessary and sufficient conditions are that $U$ leave unaltered each $\tau_i$ and that $L_1$ leave unaltered the functions

$$-\tfrac{1}{2}\xi_{3r+i} - \tfrac{1}{2}\xi_{4r+i} + \tfrac{1}{2}\xi_{5r+i} \qquad (i = 1, \cdots, r),$$

which $U$ multiplies by $-1$. Hence

$$\gamma_{2r+i,\,5r+j} = \tfrac{1}{2}\delta_{ij}, \qquad \gamma_{5r+i,\,2r+j} = \tfrac{1}{2}\delta_{ij} \qquad (i, j = 1, \cdots, r). \tag{48}$$

Then relations (45) become

$$\sum_{j=3\kappa r+1}^{m} \gamma_{2r+i,\,j}\gamma_{j,\,2r+k} = \tfrac{1}{2}\delta_{ik} \qquad (i, k = 1, \cdots, r). \tag{49}$$

In the final terms of (43) and (44) we may restrict $i$ and $j$ to the values $3\kappa r+1, \cdots, m$, provided we insert the term $\frac{1}{4}\xi_{5r+i}$ in (44), and $8\xi_{5r+l} = -\tau_i$ in (43).

We next exhibit $L_1$ in a notation better suited for the further discussion. We set

$$m = 3\kappa r + \mu, \qquad \tau_i = \tau_{i1}, \qquad \gamma_{3\kappa r+i,\,2r+j} = \tfrac{1}{2}\beta_{ij1}, \qquad \gamma_{2r+j,\,3\kappa r+i} = \tfrac{1}{2}\alpha_{j1i},$$

for $i = 1, \cdots, \mu$; $j = 1, \cdots, r$. Then

$$L_1: \quad \delta\xi_{r+i} = \delta\xi_{2r+i} = \tau_{i1}, \qquad \delta\xi_{3r+i} = \delta\xi_{5r+i} = -\tau_{i1} \qquad (i = 1, \cdots, r),$$

$$\delta\xi_{3\kappa r+i} = -\sum_{j=1}^{r} \beta_{ij1}\tau_{j1} \qquad (i = 1, \cdots, \mu),$$

$$\tau_{i1} = -\tfrac{1}{2}\xi_{r+i} - \tfrac{1}{4}\xi_{2r+i} + \tfrac{1}{2}\xi_{3r+i} + \tfrac{1}{4}\xi_{5r+i} + \tfrac{1}{4}\sum_{j=1}^{\mu} \alpha_{i1j}\xi_{3\kappa r+j} \quad (i = 1, \cdots, r),$$

$$\sum_{j=1}^{\mu} \alpha_{i1j}\beta_{jk1} = 2\delta_{ik} \qquad (i, k = 1, \cdots, r).$$

The link $[4t, 4t+1] \sim L_t$ may be obtained from $L_1$ by adding $3(t-1)r$ to the subscripts $\leqq 3\kappa r$ of the $\xi$'s. Hence, for $t = 1, \cdots, \kappa - 1$,

$$L_t: \ \delta\xi_{(3t-2)r+i} = \delta\xi_{(3t-1)r+i} = \tau_{it}, \quad \delta\xi_{3tr+i} = \delta\xi_{(3t+2)r+i} = -\tau_{it} \quad (i = 1, \cdots, r), \tag{50}$$

$$\delta\xi_{3\kappa r+j} = -\sum_{k=1}^{r} \beta_{jkt}\tau_{kt} \qquad (j = 1, \cdots, \mu),$$

$$\tau_{it} = -\tfrac{1}{2}\xi_{(3t-2)r+i} - \tfrac{1}{4}\xi_{(3t-1)r+i} + \tfrac{1}{2}\xi_{3tr+i} + \tfrac{1}{4}\xi_{(3t+2)r+i} + \tfrac{1}{4}\sum_{j=1}^{\mu} \alpha_{itj}\xi_{3\kappa r+j}, \tag{51}$$

$$\sum_{j=1}^{\mu} \alpha_{itj}\beta_{jkt} = 2\delta_{ik} \qquad (i, k = 1, \cdots, r). \tag{52}$$

Now $L_t$ and $L_s$ are commutative if and only if each leaves fixed the $\tau_i$ functions belonging to the other. According as $s = t+1$ or $s > t+1$, the condi-

tions are, respectively,

$$\sum_{j=1}^{\mu} \alpha_{it+ij}\beta_{jkt} = \delta_{ik}, \qquad \sum_{j=1}^{\mu} \alpha_{itj}\beta_{jkt+1} = \delta_{ik}, \tag{53}$$

$$\sum_{j=1}^{\mu} \alpha_{isj}\beta_{jkt} = 0, \qquad \sum_{j=1}^{\mu} \alpha_{itj}\beta_{jks} = 0 \qquad (s \geqq t+2), \tag{54}$$

holding for $s, t = 1, \cdots, \kappa - 1$; $i, k = 1, \cdots, r$.

13. The product of the two matrices

$$A_i = (\alpha_{itj}), \qquad B_k = (\beta_{jkt}) \quad (t = 1, \cdots, \kappa - 1; j = 1, \cdots, \mu), \tag{55}$$

the first having $\kappa - 1$ rows and $\mu$ columns and the second having $\mu$ rows and $\kappa - 1$ columns, is a square matrix of order $\kappa - 1$:

$$P_{ik} = (\pi_{tl}^{ik}), \qquad \pi_{tl}^{ik} \equiv \sum_{j=1}^{\mu} \alpha_{itj}\beta_{jkl} \quad (t, l = 1, \cdots, \kappa - 1). \tag{56}$$

In view of (52)–(54), we have

$$\pi_{tt}^{ik} = 2\delta_{ik}, \qquad \pi_{t+1t}^{ik} = \pi_{tt+1}^{ik} = \delta_{ik}, \qquad \pi_{st}^{ik} = \pi_{ts}^{ik} = 0 \quad (s \geqq t+2).$$

Hence if $i \neq k$, $P_{ik}$ is a matrix $O$ all of whose elements are zero. Also each $P_{ii}$ is the matrix

$$P = \begin{Vmatrix} 2 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 2 \end{Vmatrix}, \tag{57}$$

the elements in the main diagonal being 2, those in the adjacent parallels being 1, and all the remaining elements being 0. If we set $|P| = \Delta_{\kappa-1}$ and expand according to the first row, we have

$$\Delta_{\kappa-1} = 2\Delta_{\kappa-2} - \Delta_{\kappa-3},$$

whence, by induction, $\Delta_{\kappa-1} = \kappa$. As in (23), we set

$$A = \begin{Vmatrix} A_1 \\ \vdots \\ A_r \end{Vmatrix}, \qquad B = \overbrace{B_1 \cdots B_r}. \tag{58}$$

Since $A_i B_k = P_{ik}$, we have

$$AB = \begin{pmatrix} P & O & \cdots & O \\ \cdot & \cdot & \cdot & \cdot \\ O & O & \cdots & P \end{pmatrix}, \qquad |AB| = \kappa^r. \tag{59}$$

First, let $\mu < (\kappa - 1)r$. We convert matrices $A$ and $B$ into square matrices

of order $(\kappa-1)r$ by adding columns of zero elements to the former and rows of zero elements to the latter. In view of their determinants, we have $0.0=\kappa^r$. Hence the field $F$ must have a modulus which divides $\kappa$.

Next, let $\mu<(\kappa-2)r$. Drop the last row of each $A_i$ and the last column of each $B_i$. Let $A$ and $B$ then become $A^*$ and $B^*$. Hence

$$(59^*) \qquad A^*B^*=\begin{pmatrix} P^* & O & \cdots & O \\ \cdot & \cdot & \cdot & \cdot \\ O & O & \cdots & P^* \end{pmatrix}, \qquad |A^*B^*|=(\kappa-1)^r,$$

where $P^*$ is obtained from $P$ by deleting the last row and last column. Thus $|P^*|=\Delta_{\kappa-2}=\kappa-1$. We convert $A^*$ and $B^*$ into square matrices of order $(\kappa-2)r$ by adding columns of zero elements to the former and rows of zero elements to the latter. Hence $0.0=(\kappa-1)^r$, in contradiction to the preceding case.

THEOREM. *For $q\geqq 4\kappa$ we have $\mu\geqq(\kappa-1)r$ or $\mu\geqq(\kappa-2)r$, according as the field does not have or has a modulus which divides $\kappa$.*

By §§ 2–4 the symmetric group on $4\kappa$ letters can always be represented as a linear group $G$ on $4\kappa-1$ variables; and, if the modulus divides $\kappa$, as a linear group $G'$ on $4\kappa-2$ variables. By notation, $m\equiv 3\kappa r+\mu$. If $F$ does not have a modulus dividing $\kappa$, the direct product of $r$ groups $G$ gives a representation for which the number of variables is $(4\kappa-1)r=m$, with $\mu=(\kappa-1)r$. If $F$ has a modulus dividing $\kappa$, the direct product of $l$ groups $G$ and $r-l$ groups $G'$ gives a representation for which the number of variables is

$$(4\kappa-1)l+(4\kappa-2)(r-l)=m, \qquad \text{with} \qquad \mu=(\kappa-2)r+l.$$

Thus $\mu$ may have any of the values $(\kappa-2)r, (\kappa-2)r+1, \cdots$. In particular the limits which the theorem assigns to $\mu$ cannot be lowered.

The problem of the normalization of the groups is considered in §§ 22–24.

14. Let next $q\geqq 4\kappa+1$, and set $[4\kappa-1, 4\kappa+1]\sim S$. Then $S$ must be commutative with the first $\kappa-1$ triples (35) and (47), and with $\Pi C_{(3\kappa-3)r+i}$. The variables in the latter are not affected by $S$ in view of the hypothesis of § 11. Hence

$$(60) \qquad S: \qquad \xi'_{(3\kappa-2)r+i}=\sum_{j=1}^{\nu}\sigma_{ij}\,\xi_{(3\kappa-2)r+j} \qquad (i=1,\cdots,\nu),$$

where $\nu=m-(3\kappa-2)r$. Now

$$[4\kappa-1, 4\kappa]\sim P=\prod_{i=1}^{r} C_{(3\kappa-2)r+i}.$$

Hence

$$(61) \qquad [4\kappa, 4\kappa+1]\sim T=SP'S: \delta\xi_{(3\kappa-2)r+i}=-2\sum_{j=1}^{r}\sigma_{ij}\tau_j \quad (i=1,\cdots,\nu),$$

in which $\tau_i$ denotes the sum (60) for $i \leqq r$. Next, $T$ must be commutative with

$$U_{(3\kappa-3)r+1,\ldots,3\kappa r}: \quad \delta\xi_{(3\kappa-3)r+i} = \delta\xi_{(3\kappa-2)r+i} = \lambda_i, \qquad \delta\xi_{(3\kappa-1)r+i} = -2\lambda_i,$$

$$\lambda_i = -\tfrac{1}{2}\xi_{(3\kappa-3)r+i} - \tfrac{1}{2}\xi_{(3\kappa-2)r+i} + \tfrac{1}{2}\xi_{(3\kappa-1)r+i} \qquad (i=1,\cdots,r),$$

in accord with (36), (37). But $UT$ and $TU$ replace $\xi_{(3\kappa-3)r+i}$ by equal functions if and only if $T$ leaves $\lambda_i$ unaltered, the conditions for which are

$$\sigma_{ij} = \sigma_{r+ij} \qquad (i,j=1,\cdots,r). \tag{62}$$

Again, $UT$ and $TU$ replace $\xi_{(3\kappa-2)r+i}$ by equal functions if and only if $U$ leaves unaltered the sum in (61). For $i = 1, \cdots, \nu, r$ of these sums are linearly independent since the $\sigma_{ij}$ form the first $r$ columns of the matrix of $S$. Hence $U$ must leave unaltered each $\tau_j$, so that

$$\sigma_{ji} = 2\sigma_{jr+i} \qquad (i,j=1,\cdots,r). \tag{63}$$

Next, $P$ must transform $T$ into $S$. The conditions are

$$\delta_{ik} \pm 2\sum_{j=1}^{r}\sigma_{ij}\sigma_{jk} = \sigma_{ik} \qquad (i,k=1,\cdots,\nu). \tag{64}$$

the sign being minus if and only if $i$ and $k$ are both $\leqq r$ or both $> r$. Let first $k \leqq r$, $i = r+\epsilon$, $\epsilon \leqq r$; then, by (62), (64) becomes

$$2\sum_{j=1}^{r}\sigma_{\epsilon j}\sigma_{jk} = \sigma_{\epsilon k},$$

Adding this to (64) for $i = \epsilon$, we get $\frac{1}{2}\delta_{\epsilon k} = \sigma_{\epsilon k}$. Hence by (62), (63),

$$\sigma_{ij} = \sigma_{r+ij} = \tfrac{1}{2}\delta_{ij}, \qquad \sigma_{ir+j} = \tfrac{1}{4}\delta_{ij} \qquad (i,j=1,\cdots,r). \tag{65}$$

Then conditions (64) become identities unless $i > r$, $k > r$, while then the $\sigma_{ij}$ involved do not occur in (61). Hence we need not consider further the conditions (64).

To make our notations uniform with those at the end of § 12, we set

$$T = T_\kappa, \qquad \tau_i = -\tau_{i\kappa}, \qquad \sigma_{2r+j,\,k} = -\tfrac{1}{2}\beta_{jk\kappa}, \qquad \sigma_{i,\,2r+j} = -\tfrac{1}{4}\alpha_{i\kappa j}.$$

Then the preceding results give

$$T_\kappa: \ \delta\xi_{(3\kappa-2)r+i} = \delta\xi_{(3\kappa-1)r+i} = \tau_{i\kappa}, \quad \delta\xi_{3\kappa r+j} = -\sum_{k=1}^{r}\beta_{jk\kappa}\tau_{k\kappa} \quad \begin{pmatrix} i=1,\cdots,r \\ j=1,\cdots,\mu \end{pmatrix}, \tag{66}$$

$$\tau_{i\kappa} = -\tfrac{1}{2}\xi_{(3\kappa-2)r+i} - \tfrac{1}{4}\xi_{(3\kappa-1)r+i} + \tfrac{1}{4}\sum_{j=1}^{\mu}\alpha_{i\kappa j}\xi_{3\kappa r+j}. \tag{67}$$

Since $T_\kappa$ must multiply $\tau_{i\kappa}$ by $-1$, we have

$$\sum_{j\ 1}^{\mu}\alpha_{ikj}\beta_{jk\kappa} = 5\delta_{\ k} \qquad (i,k=1,\ \cdots,r). \tag{68}$$

The conditions that $T_\kappa$ be commutative with $L_{\kappa-1}$ and $L_t$ $(t<\kappa-1)$ are

$$\sum_{j=1}^{\mu}\alpha_{ikj}\beta_{jk\kappa-1}=\delta_{ik},\qquad \sum_{j=1}^{\mu}\alpha_{ik-1j}\beta_{jk\kappa}=\delta_{ik}, \tag{69}$$

$$\sum_{j=1}^{\mu}\alpha_{ikj}\beta_{jkt}=0,\qquad \sum_{j=1}^{\mu}\alpha_{itj}\beta_{jk\kappa}=0. \tag{70}$$

$\left(\begin{matrix} i,k=1,\cdots,r \\ t=1,\cdots,\kappa-2\end{matrix}\right)$

We proceed as in § 13, with an additional value $\kappa$ for $t$. Thus in (55), $A_i$ now has an additional row, $B_k$ an additional column, while in (56) and (57) there is an additional row and column, the new diagonal element being 5, the element preceding 5 being 1, that above 5 being 1, and the remaining elements being zero, by (68)–(70). Expanding $|\pi_{tt'}|$ according to the last row, we get

$$5\Delta_{\kappa-1}-\Delta_{\kappa-2}=5\kappa-(\kappa-1)=4\kappa+1.$$

THEOREM. *For $q\geqq 4\kappa+1$ we have $\mu\geqq\kappa r$ or $\mu\geqq(\kappa-1)r$, according as the field does not have or has a modulus which divides $4\kappa+1$.*

15. Let next $q\geqq 4\kappa+2$. To $[4\kappa+1,\,4\kappa+2]$ corresponds a transformation commutative with the $\kappa$ triples (35) and (47) and hence not affecting $\xi_i$ $(i=1,\cdots,3\kappa r)$. Applying a transformation on the remaining variables, we may set

$$[4\kappa+1,\,4\kappa+2]\sim Q=\prod_{i=1}^{r}C_{3\kappa r+i}. \tag{71}$$

Since the the latter must be commutative with $L_t$, we have

$$\alpha_{itj}=0,\qquad \beta_{ijt}=0\quad(i,j=1,\cdots,r\,;\,t=1,\cdots,\kappa-1). \tag{72}$$

Since $[4\kappa,\,4\kappa+1]\sim T_\kappa$, given by (66), $T_\kappa Q$ must be of period 3. The necessary and sufficient conditions are

$$\sum_{j=1}^{r}\alpha_{ikj}\beta_{jk\kappa}=2\delta_{ik}=\sum_{j=1}^{r}\alpha_{jk\kappa}\beta_{ij\kappa}\qquad(i,k=1,\cdots,r). \tag{73}$$

Thus $|\alpha|\neq 0$. We make the transformation of variables

$$\xi^*_{3\kappa r+i}=\tfrac{1}{2}\sum_{j=1}^{r}\alpha_{ikj}\xi_{3\kappa r+j}\qquad(i=1,\cdots,r).$$

Of the preceding transformations, the only one altered is $T_\kappa$. For the latter, $\delta\xi^*_{3\kappa r+i}=-\tau_{i\kappa}$. The effect of this normalization is the replacement of conditions (73) by

$$\beta_{jk\kappa}=\delta_{jk},\qquad \alpha_{jk\kappa}=2\delta_{jk}\qquad(j,k=1,\cdots,r). \tag{74}$$

The remaining parameters of our transformations define the matrices

$$A_i=(\alpha_{ikj}),\qquad B_i=(\beta_{jit})\quad(t=1,\cdots,\kappa\,;\,j=r+1,\cdots,\mu\,;\,i=1,\cdots,r). \tag{75}$$

These parameters are subject to conditions (52)–(54), (69), (70), in which, by

(72), we may now take $j = r+1, \cdots, \mu$, and to the conditions

$$\sum_{j=r+1}^{\mu} \alpha_{i\kappa j}\beta_{jk\kappa} = 3\delta_{ik} \qquad (i, k = 1, \cdots, r), \tag{76}$$

which follow from (68) and (74). The statements at the end of § 14 hold here if we replace 5 by 3; in particular, $|\pi| = 2\kappa + 1$. In applying the argument of § 13, the two cases to consider are now $\mu - r < \kappa r$ and $\mu - r < (\kappa - 1)r$.

THEOREM. *For $q \geqq 4\kappa + 2$, we have $\mu \geqq (\kappa + 1)r$ or $\mu \geqq \kappa r$, according as the field does not have or has a modulus which divides $2\kappa + 1$.*

Note that (66) and (67) simplify, in view of (74):

$$T_\kappa: \quad \delta\xi_{(3\kappa-2)r+i} = \delta\xi_{(3\kappa-1)r+i} = \tau_{i\kappa}, \qquad \delta\xi_{3\kappa r+i} = -\tau_{i\kappa} \quad (i = 1, \cdots, r), \tag{66'}$$

$$\delta\xi_{3\kappa r+j} = -\sum_{k=1}^{r} \beta_{jk\kappa}\tau_{k\kappa} \qquad (j = r+1, \cdots, \mu),$$

$$\tau_{i\kappa} = -\tfrac{1}{2}\xi_{(3\kappa-2)r+i} - \tfrac{1}{4}\xi_{(3\kappa-1)r+i} + \tfrac{1}{2}\xi_{3\kappa r+i} + \tfrac{1}{4}\sum_{j=r+1}^{\mu} \alpha_{i\kappa j}\xi_{3\kappa r+j}. \tag{67'}$$

16. Finally, let $q = 4\kappa + 3$. Since a transformation commutative with the $\kappa$ triples affects only $\xi_i$ $(i > 3\kappa r)$, we have

$$[4\kappa + 1, 4\kappa + 3] \sim E: \xi'_{3\kappa r+i} = \sum_{j=1}^{\mu} \epsilon_{ij}\xi_{3\kappa r+j} \qquad (i = 1, \cdots, \mu). \tag{77}$$

Then, by (71),

$$[4\kappa + 2, 4\kappa + 3] \sim R = EQE: \delta\xi_{3\kappa r+i} = -2\sum_{j=1}^{r} \epsilon_{ij}\omega_j \quad (i = 1, \cdots, \mu), \tag{78}$$

in which $\omega_i$ denotes the sum (77) for $i \leqq r$. Now $Q$ must transform $R$ into $E$. As in (64), the conditions are

$$\delta_{ik} \pm 2\sum_{j=1}^{r} \epsilon_{ij}\epsilon_{jk} = \epsilon_{ik} \qquad (i, k = 1, \cdots, \mu), \tag{79}$$

the sign being minus if and only if $i$ and $k$ are both $\leqq r$ or both $> r$. Set

$$e = (\epsilon_{ij}), \qquad e_\rho = e - \rho I \qquad (i, j = 1, \cdots, r). \tag{80}$$

In view of relations (79) for $i, k = 1, \cdots, r$, we have

$$e_\rho e_\sigma = (\tfrac{1}{2} + \rho\sigma)I + fe \qquad (f = -\tfrac{1}{2} - \rho - \sigma). \tag{81}$$

Set $\Delta_\rho = |e_\rho|$. Taking $f = 0$, we have

$$\Delta_\rho\Delta_{-\frac{1}{2}-\rho} = [\tfrac{1}{2} + \rho(-\tfrac{1}{2} - \rho)]^r = (1 + \rho)^r(\tfrac{1}{2} - \rho)^r. \tag{82}$$

We next prove* that $\Delta_\rho = 0$ actually has the root $\rho = \frac{1}{2}$. For a fixed value of $k$ $(k > r)$, equations (79), with $i = 1, \cdots, r$, may be written

$$\epsilon_{i1}\epsilon_{1k} + \cdots + (\epsilon_{ii} - \tfrac{1}{2})\epsilon_{ik} + \cdots + \epsilon_{ir}\epsilon_{rk} = 0. \tag{83}$$

* This is obvious if the modulus is 3, since $-1 \equiv \frac{1}{2}$.

For a fixed value of $i\,(i>r)$, equations (79), with $k=1,\cdots,r$, are

$$\epsilon_{1k}\epsilon_{i1}+\cdots+(\epsilon_{kk}-\tfrac{1}{2})\epsilon_{ik}+\cdots+\epsilon_{rk}\epsilon_{ir}=0. \tag{84}$$

Hence, if $\Delta_1\neq 0$, then $\epsilon_{ik}=0\ (i\leqq r,\ k>r)$, $\epsilon_{ik}=0\ (i>r,\ k\leqq r)$, so that $E$ would be commutative with $Q$, and the isomorphism would fail.

It follows that $\Delta_\rho=0$ has an $l$-fold root $\frac{1}{2}$ and an $(r-l)$-fold root $-1$, where $l\geqq 1$, while $l=r$ if the field has modulus 3. Applying* a transformation on the variables $\xi_{3\kappa r+i}\ (i=1,\cdots,r)$, we may, in view of the theory of canonical forms, normalize $e$ and set

$$\epsilon_{ij}=0\ (j\neq i,\ i-1),\quad \epsilon_{ii}=\tfrac{1}{2}\ (i=1,\cdots,l),\quad \epsilon_{ii}=-1\ (i=l+1,\cdots,r),\quad \epsilon_{l+1l}=0. \tag{85}$$

Then equations (79), for $i,\ k=1,\cdots,r$, reduce to

$$3\epsilon_{ii-1}=0\ (i\neq l+1),\qquad \epsilon_{ii-1}\epsilon_{i-1i-2}=0\ (i=1,\cdots,r). \tag{86}$$

Let $\kappa>0$. Then $R$ must be commutative with $T_\kappa$, given by (66′). By an argument similar to that with $U$ in § 14, the conditions are that

$$R \text{ leaves fixed each } \tau_{i\kappa},\ T_\kappa \text{ each } \omega_i \qquad (i=1,\cdots,r). \tag{87}$$

Suppose first that the field does not have modulus 3. Then by (85), (86), $\epsilon_{ii-1}=0\ (i=2,\cdots,r)$. Hence (83) and (84) reduce to

$$\epsilon_{ik}=0\ (i=l+1,\cdots,r;\ k>r),\qquad \epsilon_{ik}=0\ (i>r;\ k=l+1,\cdots,r). \tag{88}$$

Hence $E$ and $R$ are products of $\Pi\, C_{3\kappa r+i}$ by transformations not affecting

$$\xi_{3\kappa r+i} \qquad (i=l+1,\cdots,r). \tag{89}$$

Then, by† (87), $T_\kappa$ leaves unaltered the variables (89). But, for $Q$ defined by (71), $T_\kappa Q$ must be of period 3. Hence $l=r$, and

$$(\epsilon_{ij})=\tfrac{1}{2}I \qquad (i,j=1,\cdots,r). \tag{90}$$

Next let the field have modulus 3. Then $\epsilon_{ii}=-1\ (i\leqq r)$. Suppose that $\epsilon_{ii-1}\neq 0$ for a fixed value of $i$, $2\leqq i\leqq r$. Then by (86) and (83),

$$\epsilon_{i-1\,i-2}=0,\qquad \epsilon_{i-1k}=0 \qquad (k=r+1,\cdots,\mu).$$

Then $\omega_{i-1}=-\xi_{3\kappa r+i-1}$; so that, by (87), $T_\kappa$ leaves $\xi_{3\kappa r+i-1}$ unaltered. Then $T_\kappa Q$ is not of period 3, contrary to the isomorphism. Hence each $\epsilon_{ii-1}=0\ (i\leqq r)$, and relation (90) holds.

Having established (90), we introduce notations uniform with those of the preceding sections. We set

$$R=T_{\kappa+1},\quad \omega_i=-\tau_{i\kappa+1},\quad \epsilon_{ji}=-\tfrac{1}{2}\beta_{ji\kappa+1},\quad \epsilon_{ij}=-\tfrac{1}{4}\alpha_{i\kappa+1j}\quad (i\leqq r;\ j>r).$$

* If $\kappa>0$, we forego for the present the normalization (74) of $T_\kappa$.

† To give a direct proof, we note that $T_\kappa$ multiplies each variable (89) by $\pm 1$, and that $\pm 1=+1$ by the hypothesis in § 11.

Then by (78), (90), and the definition of $\omega_i$ as the sum (77),

$$(91)\quad T_{\kappa+1}\colon\ \delta\xi_{3\kappa r+i}=\tau_{i\kappa+1},\ \ \delta\xi_{3\kappa r+j}=-\sum_{k=1}^{r}\beta_{jk\kappa+1}\tau_{k\kappa+1}\quad (i=1,\cdots,r;\ j=r+1,\cdots,\mu),$$

$$(92)\qquad \tau_{i\kappa+1}=-\tfrac{1}{2}\xi_{3\kappa r+i}+\tfrac{1}{4}\sum_{j=r+1}^{\mu}a_{i\kappa+1j}\xi_{3\kappa r+j}.$$

In view of its origin, $T_{\kappa+1}$ must multiply $\tau_{i\kappa+1}$ by $-1$, whence

$$(93)\qquad \sum_{j=r+1}^{\mu}\alpha_{i\kappa+1j}\beta_{jk\kappa+1}=6\delta_{ik}\qquad (i,k=1,\cdots,r).$$

Without altering (90) we may make the normalization in § 15, by means of which $T_\kappa$ takes the form (66′). In view of (87), $T_\kappa$ and $T_{\kappa+1}$ are commutative if and only if

$$(94)\qquad \sum_{j=r+1}^{\mu}\alpha_{i\kappa+1j}\beta_{jk\kappa}=2\delta_{ik},\qquad \sum_{j=r+1}^{\mu}\alpha_{ikj}\beta_{jk\kappa+1}=2\delta_{ik}\quad (i,k=1,\cdots,r).$$

Finally, $T_{\kappa+1}$ is commutative with $L_t$, given by (50), (51), (72), if and only if

$$(95)\quad \sum_{j=r+1}^{\mu}\alpha_{i\kappa+1j}\beta_{jkt}=0,\ \sum_{j=r+1}^{\mu}\alpha_{itj}\beta_{jk\kappa+1}=0\quad (i,k=1,\cdots,r;\ t=1,\cdots,\kappa-1).$$

We proceed as at the end of § 15. We consider matrices (75) for $t=1,\cdots,\kappa+1$, viz., with an additional row in $A_i$ and an additional column in $B_i$, the new elements satisfying (93)–(95). Thus, for $i\neq k$, $A_iB_k$ is a matrix all of whose elements are zero; while

$$(96)\qquad A_iB_i=\begin{Vmatrix} 2&1&0&0&\cdots&0&0&0&0&0\\ 1&2&1&0&\cdots&0&0&0&0&0\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 0&0&0&0&\cdots&0&1&2&1&0\\ 0&0&0&0&\cdots&0&0&1&3&2\\ 0&0&0&0&\cdots&0&0&0&2&6 \end{Vmatrix},$$

a square matrix of order $\kappa+1$, whose determinant equals (§§ 13, 15)

$$6(2\kappa+1)-2^2\Delta_{\kappa-1}=2(4\kappa+3).$$

Considering as in § 13 the cases $\mu-r<(\kappa+1)r$ and $\mu-r<\kappa r$, we obtain the following

THEOREM. *For $q=4\kappa+3$, we have $\mu\geqq(\kappa+2)r$ or $\mu\geqq(\kappa+1)r$, according as the field does not have or has a modulus which divides $q$.*

17. Under the hypothesis made in § 11 that $s=0$, we have exhibited in §§ 11–16 the correspondence of generators in any simple isomorphism betewne

the symmetric group on $q$ letters and an $m$-ary linear homogeneous group in a field not having modulus 2, and have obtained lower limits on $\mu \equiv m - 3\kappa r$. The latter results may be combined into the following

THEOREM. *The minimum value of $m$ is $(q-1)r$ or $(q-2)r$, according as the field does not have or has a modulus which divides $q$.*

For $r = 1$, the hypothesis $s = 0$ is satisfied and we have the

COROLLARY. *In any representation of the symmetric group on $q$ letters as an $m$-ary linear homogeneous group, such that a transposition corresponds to a transformation conjugate with $C_1$, it is necessary that $m \geqq q-2$, while $m = q-2$ only when the field has a modulus which divides $q$.*

18. Principle of duality. If the general $m$-ary linear homogeneous group $GLH(m, F)$ in a field $F$, not having modulus 2, contains a subgroup $G_{q!}^{(r)}$ of symmetric type, such that a transposition corresponds to a transformation conjugate with $C_1 \cdots C_r$, then $GLH(m, F)$ contains a subgroup $G_{q!}^{(m-r)}$.

As this is obviously true when $q = 2$, we set $q > 2$. Then there is no invariant substitution, so that $C = C_1 \cdots C_m$ cannot occur in $G_{q!}^{(r)}$. In particular, $r < m$.

Let the even substitutions correspond to the transformations $e_1, e_2, \cdots$ of $G_{q!}^{(r)}$, and the odd substitutions to $o_1, o_2, \cdots$. Then

$$e_1,\ e_2,\ \cdots,\ o_1C,\ o_2C,\ \cdots$$

are all distinct and form an isomorphic group $G_{q!}^{(m-r)}$.

19. LEMMA. *The symmetric group on $m+3$ letters cannot be represented as an $m$-ary linear homogeneous group in a field not having modulus 2.*

The proof proceeds by induction from $m-2$ to $m$, the lemma being evidently true for $m = 1$, and true for $m = 2$ by § 17, since then $r = 1$ by § 18.

Suppose that there is a representation $G_{(m+3)!}^{(r)}$. By §§ 17, 18 we may set

$$2 \leqq r \leqq \tfrac{1}{2}m, \qquad [12] \sim C_1 \cdots C_r. \tag{97}$$

Any transformation commutative with $C_1 \cdots C_r$ is necessarily a product $S_{1,\ldots,r}\, S'_{r+1,\ldots,m}$. Hence by § 7 the symmetric group on the $m+1$ letters $3, \cdots, m+3$ is simply isomorphic with a linear homogeneous group on $r$ or $m-r$ variables. By $(97_1)$, $r \leqq m-2$, $m-r \leqq m-2$. Hence the hypothesis for the induction is contradicted.

20. Suppose, finally, that the symmetric group on $m+2$ letters is representable as an $m$-ary linear homogeneous group in a field $F$ not having modulus 2. If $r = 1$ or $m-1$, $F$ must have a modulus which divides $m+2$ (§ 17). If $1 < r < m-1$, we may assume that (97) holds. As in § 19, the symmetric group $G$ on the $m$ letters $3, \cdots, m+2$ is simply isomorphic with a group on $r$

or else $m-r$ variables; while by the lemma there proved the number of variables is $\geqq m-2$. For either alternative, it follows from $(97_1)$ that $r=2$. Then $m \geqq 4$, so that (§ 4 or §§ 17–19) $G$ cannot be represented on $r=2$ variables.

Hence by the proof given in § 7, there remain only two possibilities:

(i) The group $G$ affects only the variables $\xi_3, \cdots, \xi_m$;

(ii) The transformations of $G$ corresponding to even substitutions affect only the variables $\xi_3, \cdots, \xi_m$; while those corresponding to odd substitutions are products of transformations on $\xi_3, \cdots, \xi_m$ by a fixed transformation $T$, of period 2, on $\xi_1$ and $\xi_2$.

For case (i), we have (18) with $r=2$, $s=0$. Hence by § 17, the minimum number of variables in a representation of $S_{(m+2)!}$ is $2m$. We may proceed directly and conclude from $[12]\sim C_1C_2$, $[34]\sim C_3C_4$, $[56]\sim C_5C_6$, $\cdots$ that at least $m+2$ or $m+1$ variables are necessary, according as $m+2$ is even or odd.

For case (ii), we apply a transformation on $\xi_1, \xi_2$, and set $T=C_1$, since obviously $T \neq C_1C_2$. Applying a transformation on $\xi_3, \cdots, \xi_m$, we may set $[34]\sim C_1C_3$, $[56]\sim C_1C_4$, and, as before, $[12]\sim C_1C_2$. The case thus falls under § 9 with $r=2$, $s=1$. Then by (20) we may set $[14][23]\sim(\xi_2\xi_3)C_2C_3C_k$, where $k=1$ if $\sigma=1$, $k=4$ if $\sigma=0$. Let $[23]\sim W$. Since $[23]$ is commutative with $[56]$ and $[14][23]$, we have $W=(a)_1(b)_4S$, $S$ affecting only $\xi_2, \xi_3, \xi_5, \cdots$. Since $[12][23]$ is of period 3, we have $a=-1$. If $b=-1$, $[23]$ and $[56]$ both correspond to $C_1C_4$. Hence $W=C_1S$. It follows that $S_{(m+2)!}$ is represented as a subgroup of the direct product of $\{I, C_1\}$ and a group on $\xi_2, \cdots, \xi_m$, and hence (§ 7) is simply isomorphic with a group on $m-1$ variables. But this contradicts § 19.

Hence cases (i) and (ii) are both excluded.

THEOREM. *If the symmetric group on $q$ letters is representable as a linear homogeneous group on $q-2$ variables in a field $F$ not having modulus 2, then $r=1$ or $q-3$, and $F$ has a modulus which divides $q$.*

21. In view of §§ 8, 19, 20, we may state the complete

THEOREM. *Let $F$ be an arbitrary field. Then the minimum number of variables for the representation of the symmetric group on $q$ letters is $q-1$ or $q-2$, according as $F$ has not or has a modulus which divides $q$. In the latter case exceptions arise for $q \leqq 4$; when the modulus divides $q$, the minimum number of variables is 2 for $q=2$ or 3, and 3 for $q=4$.*

22. In order to obtain a clear insight into the nature of the linear groups exhibited in §§ 11–16, we shall make a normalization by a transformation of variables which does not disturb the correspondences (35) and (47). To normalize the $L_t$, given by (50), set

$$\eta_{3\kappa r+i} = \sum_{j=1}^{\mu} \lambda_{ij}\, \xi_{3\kappa r+j} \qquad (i=1, \cdots, \mu). \tag{98}$$

Then, under $L_t$,

$$\delta\eta_{3\kappa r+i} = -\sum_{k=1}^{r}\beta'_{ikt}\tau_{kt}, \qquad \beta'_{ikt} = \sum_{j=1}^{\mu}\lambda_{ij}\beta_{jkt}. \tag{99}$$

Suppose first that the field does not have a modulus which divides $\kappa$. Then, by § 13, $\mu \geqq (\kappa-1)r$. As in (55) and (58), set

$$B'_k = (\beta'_{jkt}), \qquad B' = \widehat{B'_1 \cdots B'_r} \quad (t=1,\cdots,\kappa-1;\ j=1,\cdots,\mu). \tag{100}$$

Thus $(\lambda)B = B'$. Now $|AB|$ equals the sum of all products of the determinants of order $(\kappa-1)r$ formed from $(\kappa-1)r$ columns of $A$ by the corresponding determinants formed from $(\kappa-1)r$ rows of $B$. But $|AB| \neq 0$ by (59). Hence, after making a suitable permutation of the variables $\xi_{3\kappa r+j}$, we may set $|\bar{B}| \neq 0$, where

$$\bar{B} = \widehat{\bar{B}_1 \cdots \bar{B}_r}, \qquad \bar{B}_k = (\beta_{jkt}) \quad [t=1,\cdots,\kappa-1;\ j=1,\cdots,(\kappa-1)r], \tag{101}$$

so that $\bar{B}_k$ is composed of the first $(\kappa-1)r$ rows of $B_k$. For the purpose of assigning an order to the $\beta$'s in the $j$th row of $\bar{B}$, we set

$$\beta_{jkt} \equiv \beta_{j,(\kappa-1)(k-1)+t}. \tag{102}$$

For a fixed value $\leqq \mu$ of $i$, the equations

$$\beta'_{ikt} = \delta_{i,(\kappa-1)(k-1)+t} \quad (t=1,\cdots,\kappa-1;\ k=1,\cdots,r) \tag{103}$$

uniquely determine $\lambda_{i1},\cdots,\lambda_{i,(\kappa-1)r}$ as linear functions of the remaining $\lambda_{ij}$, since the determinant of the coefficients of the former equals $|\bar{B}|$. Let $(\lambda)$ denote the matrix in (98) and

$$L = (\lambda_{ij}) \quad [i=1,\cdots,\mu;\ j=(\kappa-1)r+1,\cdots,\mu]. \tag{104}$$

We have the following relation between matrices of order $\mu$:

$$(\lambda)(\widehat{B\Omega}) = (\widehat{B'L}), \tag{105}$$

where $\Omega$ is a matrix of $\mu$ rows and $\mu-(\kappa-1)r$ columns, all of whose elements are zero, except those lying in the main diagonal of $\widehat{B\Omega}$ which equal unity, i. e.,

$$\Omega = (\delta_{i,(\kappa-1)r+j}) \quad [i=1,\cdots,\mu;\ j=1,\cdots,\mu-(\kappa-1)r]. \tag{106}$$

Hence by (103) and (105),

$$B' = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ . & . & . & . \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ . & . & . & . \end{bmatrix}, \qquad |\widehat{B'L}| = |\lambda_{ij}| \quad [i,j=(\kappa-1)r+1,\cdots,\mu]. \tag{107}$$

Hence, by (105), if we take the last determinant different from zero, we have $|\lambda| \neq 0$ in (98).

We assume that this preliminary normalization has been made, so that in $L_t$ matrix $B$ has the form $(107_1)$. Let

$$A_i = (\alpha_{itj}), \qquad \bar{A} = \left.\begin{matrix} A_1 \\ \vdots \\ \bar{A}_r \end{matrix}\right\} \quad [t=1, \cdots, \kappa-1\,;\, j=1, \cdots, (\kappa-1)r], \tag{108}$$

so that $A_i$ is composed of the first $(\kappa-1)r$ columns of $A_i$. Then $\bar{A} = \bar{A}B = AB$, so that $\bar{A} = M$, where $M$ is the matrix displayed in (59). Set

$$a_i = (\kappa-1)(i-1) \qquad (i=1, \cdots, r). \tag{109}$$

Then the final equations in (50) may be replaced by

$$\delta\xi_{3\kappa r+a_i+t} = -\tau_{it} \qquad (i=1, \cdots, r).$$

In (51) the terms of the sum given by $j=1, \cdots, (\kappa-1)r$ reduce to

$$\tfrac{1}{4}(\xi_{3\kappa r+a_i+t-1} + 2\xi_{3\kappa r+a_i+t} + \xi_{3\kappa r+a_i+t+1}),$$

subject to the restrictions noted on (113). Apply the transformation of variables

$$\eta_{3\kappa r+i} = \xi_{3\kappa r+i} + \sum_{j=(\kappa-1)r+1}^{\mu} \lambda_{ij}\xi_{3\kappa r+j} \qquad [i=1, \cdots, (\kappa-1)r], \tag{110}$$

from which $\xi = \eta - \sum\lambda\eta$. Then $L_t$ is altered in form and

$$\tau_{it}(\xi) = \tau_{it}(\eta) - \tfrac{1}{4}\sum_{j=(\kappa-1)r+1}^{\mu} \alpha'_{itj}\eta_{3\kappa r+j},$$

$$\alpha'_{itj} \equiv \lambda_{a_i+t-1,j} + 2\lambda_{a_i+t,j} + \lambda_{a_i+t+1,j}.$$

Hence $(\alpha') = M(\lambda_{ij})$, where $(\lambda_{ij})$ is the matrix in (110). Taking $(\lambda) = M^{-1}(\alpha)$, we have $(\alpha') = (\alpha)$. Hence the effect of this normalization is to set $\alpha_{itj} = 0$ for $j > (\kappa-1)r$. Finally, we replace

$$\xi_{3\kappa r+a_i+t} \text{ by } \xi_{3\kappa r+(t-1)r+i} \quad (i=1, \cdots, r\,;\, t=1, \cdots, \kappa-1), \tag{111}$$

noting that $a_i + t = a_k + t'$ implies $i = k$, $t = t'$. Hence

$$L_t\text{: first four equations as in (50)},\ \delta\xi_{(3\kappa+t-1)r+i} = -\tau_{it} \quad (i=1, \cdots, r), \tag{112}$$

where $\tau_{it}$ is given by (51) with the final sum replaced by

$$\tfrac{1}{4}(\xi_{(3\kappa+t-2)r+i} + 2\xi_{(3\kappa+t-1)r+i} + \xi_{(3\kappa+t)r+i}), \tag{113}$$

the first term being absent if $t=1$, the third if $t=\kappa-1$. We have now proved, for $q=4\kappa$, the following

THEOREM. *For any field $F$ having neither modulus 2 nor a modulus which divides $q$, all linear homogeneous groups in $F$, which are simply isomorphic with the symmetric group on $q$ letters and which satisfy the condition*

*$s = 0$ imposed in § 11, are equivalent under linear transformation in $F$ to a direct product of $r$ linear groups, with fixed coefficients cogredient in the $r$ sets of variables*

$$(114) \qquad \xi_i,\ \xi_{r+i},\ \xi_{2r+i},\ \cdots,\ \xi_{(q-2)r+i} \qquad (i=1,\cdots,r).$$

To prove the theorem for the remaining cases, we normalize $T_\kappa$ and $T_{\kappa+1}$ simultaneously with the $L_t$.

First, let $q = 4\kappa + 1$. For $T_\kappa$, given by (66), we have (99) with $t = \kappa$. The above proof is to be modified by changing $\kappa - 1$ to $\kappa$, allowing $t$ to take the additional value $\kappa$, and by taking as $M$ the matrix defined at the end of § 14. We find that $L_t$ is given by (112), with no suppression in $\tau_{it}$ when $t = \kappa - 1$, and that

$$(115) \quad T_\kappa\colon\ \delta\xi_{(3\kappa-2)r+i} = \delta\xi_{(3\kappa-1)r+i} = \tau_{i\kappa}, \qquad \delta\xi_{(4\kappa-1)r+i} = -\tau_{i\kappa} \quad (i=1,\cdots,r),$$

$$(116) \qquad \tau_{i\kappa} = -\tfrac{1}{2}\xi_{(3\kappa-2)r+i} - \tfrac{1}{4}\xi_{(3\kappa-1)r+i} + \tfrac{1}{4}\xi_{(4\kappa-2)r+i} + \tfrac{5}{4}\xi_{(4\kappa-1)r+i}.$$

For $q = 4\kappa + 2$, we apply transformation (99) with $i, j = r+1, \cdots, \mu$, and modify the proof in the light of § 15. In particular, $t = 1, \cdots, \kappa$, and $j > r$ in (100), (101). Hence

$$(117) \quad L_t\colon \text{ first four equations as in (50)}, \qquad \delta\xi_{(3\kappa+t)r+i} = -\tau_{it} \quad (i=1,\cdots,r),$$

where $\tau_{it}$ is given by (51) with the final sum replaced by

$$(118) \qquad \tfrac{1}{4}(\xi_{(3\kappa+t-1)r+i} + 2\xi_{(3\kappa+t)r+i} + \xi_{(3\kappa+t+1)r+i}),$$

the first term being absent if $t = 1$. Furthermore,

$$(119) \quad T_\kappa\colon \text{ first three equations as in (66')}, \qquad \delta\xi_{4\kappa r+i} = -\tau_{i\kappa} \quad (i=1,\cdots,r),$$

$$(120) \quad \tau_{i\kappa} = -\tfrac{1}{2}\xi_{(3\kappa-2)r+i} - \tfrac{1}{4}\xi_{(3\kappa-1)r+i} + \tfrac{1}{2}\xi_{3\kappa r+i} + \tfrac{1}{4}\xi_{(4\kappa-1)r+i} + \tfrac{3}{4}\xi_{4\kappa r+i}.$$

Finally, for $q = 4\kappa + 3$, the proof is quite similar to that in the preceding case. Here $t$ takes the further value $\kappa + 1$. We obtain the same normalized forms for $L_r$ and $T_\kappa$, except that now

$$(121) \qquad \tau_{i\kappa} = \tfrac{1}{2}\xi_{(4\kappa+1)r+i} + \text{the five terms of (120)}.$$

Furthermore,

$$(122) \qquad T_{\kappa+1}\colon\ \delta\xi_{3\kappa r+i} = \tau_{i\kappa+1}, \qquad \delta\xi_{(4\kappa+1)r+i} = -\tau_{i\kappa+1} \quad (i=1,\cdots,r),$$

$$(123) \qquad \tau_{i\kappa+1} = -\tfrac{1}{2}\xi_{3\kappa r+i} + \tfrac{1}{2}\xi_{4\kappa r+i} + \tfrac{3}{2}\xi_{(4\kappa+1)r+i}.$$

23. Since the linear groups exhibited in § 22 are direct products of $r$ similar groups, it suffices to know the character of the component groups. We shall list the quadratic invariants of the groups for the case $r = 1$, assuming still that the field has neither modulus 2, nor a modulus dividing $q$. Set

$$(124) \quad f = \xi_1^2 + \xi_2^2 + \tfrac{1}{2}\xi_3^2 + \xi_4^2 + \xi_5^2 + \tfrac{1}{2}\xi_6^2 + \cdots + \xi_{3\kappa-2}^2 + \xi_{3\kappa-1}^2 + \tfrac{1}{2}\xi_{3\kappa}^2.$$

Then the only quadratic invariant of the group is

$$f+\xi_{3\kappa+1}^2+\xi_{3\kappa+1}\xi_{3\kappa+2}+\xi_{3\kappa+2}^2+\xi_{3\kappa+2}\xi_{3\kappa+3}+\xi_{3\kappa+3}^2+\cdots+\xi_{4\kappa-2}^2+\xi_{4\kappa-2}\xi_{4\kappa-1}+\xi_{4\kappa-1}^2 \qquad (q=4\kappa), \tag{125}$$

$$\text{terms of (125)}+\xi_{4\kappa-1}\xi_{4\kappa}+\tfrac{1}{2}\xi_{4\kappa}^2 \qquad (q=4\kappa+1), \tag{126}$$

$$f+\xi_{3\kappa+1}^2+\xi_{3\kappa+2}^2+\xi_{3\kappa+2}\xi_{3\kappa+3}+\xi_{3\kappa+3}^2+\cdots+\xi_{4\kappa-1}^2+\xi_{4\kappa-1}\xi_{4\kappa}+\xi_{4\kappa}^2+\xi_{4\kappa}\xi_{4\kappa+1}+\tfrac{3}{2}\xi_{4\kappa+1}^2 \qquad (q=4\kappa+2), \tag{127}$$

$$\text{terms of (127)}+2\xi_{4\kappa+1}\xi_{4\kappa+2}+3\xi_{4\kappa+2}^2 \qquad (q=4\kappa+3). \tag{128}$$

The discriminant of the quadratic invariant is $2^l|P|$, where $l=2\kappa$ in the first two cases and $l=2\kappa+1$ in the last two; while, for $q=4\kappa$, $P$ is the matrix (57) of order $\kappa-1$; for $q=4\kappa+1$ and $4\kappa+2$, $P$ is the matrix of order $\kappa$ defined at the end of §§ 14 and 15; and for $q=4\kappa+3$, $P$ is the matrix (96) of order $\kappa+1$. The matrix of the discriminant therefore reflects an essential property of the group. In each case the discriminant is not zero.

24. In conclusion, we consider briefly the normalization of our linear groups for the case of fields having a modulus $p\,(p>2)$ which divides $q$. It is no longer true that, for a given value of $r$, the groups are all equivalent under linear transformation. For simplicity, we consider only the case in which the number of variables is the minimum $(q-2)r$.

Let first $q=4\kappa$. Then by hypothesis $\mu=(\kappa-2)r$, and the modulus divides $\kappa$. Matrices $A^*$ and $B^*$, defined at the end of § 13, are now square matrices of order $\mu=(\kappa-2)r$, and each has its determinant $\neq 0$ by (59*). Hence in (98) and (99) we may take $(\lambda_{ij})=(B^*)^{-1}$. After this normalization, we have $B^*=I$. Now by (59),

$$AB^*=N=\begin{pmatrix} P' & O & \cdots & O \\ \cdot & \cdot & \cdots & \cdot \\ O & O & \cdots & P' \end{pmatrix},$$

where $P'$ is derived from $P$, given by (57), by deleting the last column. Hence $A=N$. The $\beta_{jk\kappa-1}$ are now uniquely determined by (59). Set

$$\beta_{(\kappa-2)(i-1)+v,\,k,\,\kappa-1}=b_v \qquad (i,k=1,\cdots,r;\ v=1,\cdots,\kappa-2).$$

Then

$$2b_1+b_2=0, \qquad b_1+2b_2+b_3=0, \cdots, \qquad b_{v-1}+2b_v+b_{v+1}=0, \cdots,$$
$$b_{\kappa-3}+2b_{\kappa-2}=\delta_{ik}, \qquad b_{\kappa-2}=2\delta_{ik}.$$

Since the eliminant of these $\kappa-1$ equations is $|P|$, a multiple of the modulus, there is a unique set of solutions:

$$b_v=(-1)^{\kappa-v}(\kappa-v)\delta_{ik}.$$

Proceeding similarly when $q = 4\kappa + 1, \cdots, 4\kappa + 3$, we obtain the

THEOREM. *When the field $F$ has a modulus $p\,(p > 2)$ which divides $q$, all linear homogeneous groups, which are simply isomorphic with the symmetric groups on $q$ letters, and which affect the minimum number $(q-2)r$ of variables under the hypothesis of* § 11, *are equivalent under linear transformation in $F$ to a direct product of $r$ cogredient linear groups with fixed coefficients.*

It is sufficient to exhibit one of the component groups. Hence we set $r = 1$.

For $q = 4\kappa$, $L_t\,(t \leqq \kappa - 2)$ is defined by (112) for $r = i = 1$, with the first term of (113) deleted if $t = 1$, and the last term if $t = \kappa - 2$. Also

$$(129)\quad L_{\kappa-1}\colon\ \delta\xi_{3\kappa-4}=\delta\xi_{3\kappa-3}=\tau_{\kappa-1},\quad \delta\xi_{3\kappa-2}=\delta\xi_{3\kappa}=-\tau_{\kappa-1},\quad \delta\xi_{3\kappa+j}=(-1)^{\kappa-j}j\tau_{\kappa-1}\quad (j=1,\cdots,\kappa-2),$$

$$(130)\qquad \tau_{\kappa-1}=-\tfrac{1}{2}\xi_{3\kappa-4}-\tfrac{1}{4}\xi_{3\kappa-3}+\tfrac{1}{2}\xi_{3\kappa-2}+\tfrac{1}{4}\xi_{3\kappa}+\tfrac{1}{4}\xi_{4\kappa-2}.$$

For $q = 4\kappa + 1$, $L_t$ is given by (112) for $r = i = 1$, with the suppressions noted on (113), while

$$(131)\quad T_\kappa\colon\ \delta\xi_{3\kappa-1}=\delta\xi_{3\kappa}=\tau_\kappa,\qquad \delta\xi_{3\kappa+j}=-(-1)^{\kappa-j}4j\tau_\kappa\quad (j=1,\cdots,\kappa-1),$$

$$(132)\qquad \tau_\kappa=-\tfrac{1}{2}\xi_{3\kappa-1}-\tfrac{1}{4}\xi_{3\kappa}+\tfrac{1}{4}\xi_{4\kappa-1}.$$

For $q = 4\kappa + 2$, $L_t$ is given by (117) for $r = i = 1$, with the first term of (118) suppressed if $t = 1$, and the last term if $t = \kappa - 1$. Also

$$(133)\quad T_\kappa\colon \delta\xi_{3\kappa-1}=\delta\xi_{3\kappa}=\tau_\kappa,\ \delta\xi_{3\kappa+1}=-\tau_\kappa,\ \delta\xi_{3\kappa+j}=(-1)^{\kappa-j}(2j-2)\tau_\kappa\quad (j=2,\cdots,\kappa),$$

$$(134)\qquad \tau_\kappa=-\tfrac{1}{2}\xi_{3\kappa-1}-\tfrac{1}{4}\xi_{3\kappa}+\tfrac{1}{2}\xi_{3\kappa+1}+\tfrac{1}{4}\xi_{4\kappa}.$$

Finally, for $q = 4\kappa + 3$, $L_t$ is given by (117), with the first term of (118) suppressed if $t = 1$; $T_\kappa$ is given by (119), while

$$(135)\quad T_{\kappa+1}\colon\ \delta\xi_{3\kappa+1}=\tau_{\kappa+1},\qquad \delta\xi_{2\kappa+j}=-(-1)^{\kappa-j}(4j-4)\tau_{\kappa+1}\quad (j=2,\cdots,\kappa+1),$$

$$(136)\qquad \tau_{\kappa+1}=-\tfrac{1}{2}\xi_{3\kappa+1}+\tfrac{1}{2}\xi_{4\kappa+1}.$$

We may now show that in the present case for which the modulus divides $q$, the quadratic invariant of the group is obtained from the invariant when the modulus is prime to $q$ by suppressing the final two terms. For instance, if $q = 4\kappa + 1$ the invariant is (125), if $q = 4\kappa + 3$ the invariant is (127).

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# SURFACES WITH ISOTHERMAL REPRESENTATION OF THEIR LINES OF CURVATURE AND THEIR TRANSFORMATIONS*

BY

LUTHER PFAHLER EISENHART

## INTRODUCTION.

In 1897 THYBAUT† showed that minimal surfaces admit of transformations which are very similar to the BÄCKLUND transformations of pseudospherical surfaces. The given minimal surface $\bar{S}$ and the transform $\bar{S}_1$ are the focal sheets of a $W$-congruence. This transformation is such that the minimal surfaces $S$ and $S_1$ adjoint to $\bar{S}$ and $\bar{S}_1$ respectively are the sheets of an envelope of spheres whose centers lie on a surface applicable to a paraboloid of revolution, by the theorem of GUICHARD. DARBOUX‡ and BIANCHI§ have considered the pairs of isothermic surfaces which are at the same time in conformal correspondence and form the envelope of a family of spheres, and have thus established a transformation from a given isothermic surface to another isothermic surface which together form such a pair. BIANCHI has shown that these transformations admit of a theorem of permutability very similar to the theorem of this kind which obtains for the BÄCKLUND transformations.

In the present paper we shall show that there is a transformation of surfaces with isothermal spherical representation of their lines of curvature, changing such a surface into one of the same kind. When in particular the given surface is minimal its transform is minimal and the surfaces are in the relation of $S$ and $S_1$ mentioned above.

It is known that if tangents be drawn to a surface and perpendicular to the direction of an infinitesimal deformation of the surface, these tangents form a $W$-congruence for which the given surface is one of the focal sheets. If the given surface be minimal and it be required that the lines of curvature correspond on the two surfaces, the second focal surface also is a minimal surface.

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† *Sur la déformation du paraboloide*, etc., Annales de l' École Normale (3), vol. 14 (1897), pp. 45–96.

‡ *Sur les surfaces isothermiques*, Annales de l' École Normale (3), vol. 16 (1899), pp. 491–508.

§ *Ricerche sulle superficie isoterme e sulla deformazione delle quadriche*, Annali di Matematica (3), vol. 11 (1905), pp. 93–157.

In this way we get the THYBAUT transformations of minimal surfaces and obtain the equations of condition on two functions $w$ and $\phi$ and a constant $m$, which determine the transformation. The relation between the two surfaces being perfectly reciprocal, the first is a transform of the second and the transformation functions are expressible simply in terms of $m$, $\phi$, $w$.

We have remarked that the minimal surfaces $S$ and $S_1$, adjoint to the minimal surfaces $\bar{S}$ and $\bar{S}_1$ which are THYBAUT transforms of one another, are the sheets of an envelope of spheres whose centers lie on a surface applicable to a paraboloid of revolution. By means of a theorem of MOUTARD we show in § 4 that with each surface whose lines of curvature have isothermal spherical representation there are associated $\infty^4$ surfaces of the same kind, each of which forms with the given one the envelope of a family of spheres depending upon two parameters. The determination of these new surfaces is the same problem as finding the functions $m$, $\phi$, $w$ of a THYBAUT transformation and quadratures. We call the surfaces under discussion the surfaces $\Sigma$.

The transformations which we have discovered possess the following *theorem of permutability: If a surface* $\Sigma$ *be transformed into two surfaces* $\Sigma_1$ *and* $\Sigma_2$ *of the same kind by means of transformations involving the constants* $m_1$ *and* $m_2$ *respectively, there exists a surface* $\Sigma'$ *which is the transform of* $\Sigma_1$ *and* $\Sigma_2$ *by means of transformations involving* $m_2$ *and* $m_1$ *respectively; and this surface can be found without quadrature.*

We shall say that these four surfaces $\Sigma$, $\Sigma_1$, $\Sigma_2$, $\Sigma'$ form a *quatern.* These transformations of the surfaces $\Sigma$ carry with them a transformation of the surfaces of centers of the spheres enveloped by the several pairs of surfaces $\Sigma$; and these latter transformations evidently possess a theorem of permutability similar to the above. In § 7 it is shown that if four surfaces $\Sigma$, $\Sigma_1$, $\Sigma_2$, $\Sigma'$, form a quatern involving the constants $m_1$, $m_2$, each can be transformed by means of a transformation involving a third constant $m_3$ in such a way that the four new surfaces also form a quatern.

In § 8 we call attention to the fact that surfaces with plane lines of curvature in both systems are surfaces $\Sigma$ and show that the determination of their transformations reduces to quadratures. When these transforms also have plane lines of curvature in both systems, the surfaces of centers of the spheres enveloped by pairs of them are surfaces of translation whose generators are in perpendicular planes. Moreover, the cyclides of DUPIN play an important rôle in the theory.

We close with a discussion of the surfaces $\Sigma$ with spherical lines of curvature in one system. In particular, we find that when the curves $v = \text{const.}$ of a minimal surface are spherical there can be found by quadratures an infinity of surfaces $\Sigma$ for which the curves $v = \text{const.}$ are spherical and another infinity for which the curves $u = \text{const.}$ are spherical.

## § 1. *Transformation of Thybaut.*

Let $S$ be a minimal surface referred to its lines of curvature, and let the parameters be so chosen that the linear element of the surface and of its spherical representation can be written *

$$ds^2 = e^{2\theta}(du^2 + dv^2), \tag{1}$$

and

$$ds'^2 = e^{-2\theta}(du^2 + dv^2) \tag{2}$$

respectively. Now $\theta$ is a solution of the equation

$$\frac{\partial^2\theta}{\partial u^2} + \frac{\partial^2\theta}{\partial v^2} = e^{-2\theta}. \tag{3}$$

Moreover, every solution of this equation gives a minimal surface.

From (1) and (2) it follows that the second fundamental coefficients of the surface are

$$D = -D'' = -1, \qquad D' = 0. \tag{4}$$

Denote by $X_1, Y_1, Z_1$; $X_2, Y_2, Z_2$; $X, Y, Z$; the direction cosines of the tangents to the curves $v = \text{const.}$, $u = \text{const.}$ on the surface and of its normal. Thus

$$X_1 = e^{-\theta}\frac{\partial x}{\partial u}, \qquad X_2 = e^{-\theta}\frac{\partial x}{\partial v}. \tag{5}$$

From these are found readily †

$$\begin{aligned} &\frac{\partial X_1}{\partial u} = -\frac{\partial\theta}{\partial v}X_2 - e^{-\theta}X, \qquad \frac{\partial X_2}{\partial u} = \frac{\partial\theta}{\partial v}X_1, \qquad \frac{\partial X}{\partial u} = e^{-\theta}X_1, \\ &\frac{\partial X_1}{\partial v} = \frac{\partial\theta}{\partial u}X_2, \qquad \frac{\partial X_2}{\partial v} = -\frac{\partial\theta}{\partial u}X_1 + e^{-\theta}X, \qquad \frac{\partial X}{\partial v} = -e^{-\theta}X_2, \end{aligned} \tag{6}$$

and similar expressions in the $Y$ and $Z$.

The minimal surface $\bar{S}$ adjoint to $S$ is given by quadratures of the form ‡

$$\frac{\partial\bar{x}}{\partial u} = \frac{\partial x}{\partial v}, \qquad \frac{\partial\bar{x}}{\partial v} = -\frac{\partial x}{\partial u}. \tag{7}$$

The linear element of $\bar{S}$ and of its spherical representation are the same as for $S$ and the second fundamental coefficients have the values

$$\bar{D} = \bar{D}'' = 0, \qquad \bar{D}' = 1. \tag{8}$$

Hence the parametric curves on $\bar{S}$ are its asymptotic lines.

* BIANCHI, *Lezioni*, II, p. 335.

† Ibid., I, p. 123; German translation, p. 94.

‡ Ibid., II, p. 336.

Denote by $w$ the characteristic function in an infinitesimal deformation of $\bar{S}$; then $w$ is any solution of the equation

$$\frac{\partial^2 w}{\partial u \partial v} + \frac{\partial \theta}{\partial v}\frac{\partial w}{\partial u} + \frac{\partial \theta}{\partial u}\frac{\partial w}{\partial v} = 0. \tag{9}$$

The direction-cosines of the direction of deformation of $\bar{S}$ are proportional to the coördinates, $x_0$, $y_0$, $z_0$, of a surface $S_0$. When a solution $w$ of equation (9) is known, these coördinates are given by the quadratures †

$$\frac{\partial x_0}{\partial u} = e^{\theta}\left(wX_1 - e^{\theta}\frac{\partial w}{\partial u}X\right), \qquad \frac{\partial x_0}{\partial v} = e^{\theta}\left(wX_2 + e^{\theta}\frac{\partial w}{\partial v}X\right). \tag{10}$$

If lines be drawn tangent to $\bar{S}$ and perpendicular to the direction of the infinitesimal deformation of $\bar{S}$, they form a $W$-congruence, for which $\bar{S}$ is one of the focal sheets. Moreover, this is a general construction for $W$-congruences. ‡

Denote by $\omega$ the angle which the line so drawn at a point of $\bar{S}$ makes with the tangent to the curve $v = \text{const.}$ at the point; then

$$\Sigma x_0(X_1 \cos\omega + X_2 \sin\omega) = 0,$$

or

$$\cos\omega : \sin\omega = \Sigma x_0 X_2 : -\Sigma x_0 X_1. \tag{11}$$

We shall put this result in a more suitable form.

We introduce the function $T$ defined by

$$T = \Sigma X x_0, \tag{12}$$

and in accordance with equation (9) we put

$$\frac{\partial \psi}{\partial u} = e^{2\theta}\frac{\partial w}{\partial u}, \qquad \frac{\partial \psi}{\partial v} = -e^{2\theta}\frac{\partial w}{\partial v}. \tag{13}$$

From (12) we get by differentiation

$$\frac{\partial T}{\partial u} = e^{-\theta}\Sigma x_0 X_1 - \frac{\partial \psi}{\partial u}, \qquad \frac{\partial T}{\partial v} = -e^{-\theta}\Sigma x_0 X_2 - \frac{\partial \psi}{\partial v}. \tag{14}$$

It is readily found that

$$\frac{\partial}{\partial v}(e^{\theta}\Sigma x_0 X_1) = \frac{\partial}{\partial u}(e^{\theta}\Sigma x_0 X_2).$$

In consequence of this equation we define a function $\phi$ so that

$$\frac{\partial \phi}{\partial u} = me^{\theta}\Sigma x_0 X_1, \qquad \frac{\partial \phi}{\partial v} = me^{\theta}\Sigma x_0 X_2, \tag{15}$$

* Ibid., II, p. 7; German translation, p. 291.

† Ibid., p. 6; German translation, p. 290.

‡ Ibid., II, pp. 52, 53; German translation, p. 316.

where $m$ is an arbitrary constant. With the help of (6) and (10) the second differential coefficients of $\phi$ are reducible to

$$\begin{aligned}
\frac{\partial^2\phi}{\partial u^2} &= \frac{\partial\theta}{\partial u}\frac{\partial\phi}{\partial u} - \frac{\partial\theta}{\partial v}\frac{\partial\phi}{\partial v} - mT + me^{2\theta}w,\\
\frac{\partial^2\phi}{\partial u\partial v} &= \frac{\partial\theta}{\partial v}\frac{\partial\phi}{\partial u} + \frac{\partial\theta}{\partial u}\frac{\partial\phi}{\partial v},\\
\frac{\partial^2\phi}{\partial v^2} &= -\frac{\partial\theta}{\partial u}\frac{\partial\phi}{\partial u} + \frac{\partial\theta}{\partial v}\frac{\partial\phi}{\partial v} + mT + me^{2\theta}w.
\end{aligned} \tag{16}$$

We return to the consideration of the congruence of tangents to $\bar{S}$. The coördinates of a point on the tangent at the point $(\bar{x}, \bar{y}, \bar{z})$ are, in consequence of (11) and (15), of the form

$$\bar{x}_1 = \bar{x} + \frac{\lambda e^{-\theta}}{m}\left(\frac{\partial\phi}{\partial v}X_1 - \frac{\partial\phi}{\partial u}X_2\right), \tag{17}$$

where $\lambda$ is a function of $u$ and $v$. Differentiating with respect to $u$ and $v$, we get

$$\begin{aligned}
\frac{\partial\bar{x}_1}{\partial u} &= [e^{-\theta}\lambda T + e^{\theta}(1-\lambda w)]X_2 - e^{-2\theta}\frac{\lambda}{m}\frac{\partial\phi}{\partial v}X + \frac{e^{-\theta}}{m}\left(\frac{\partial\phi}{\partial v}X_1 - \frac{\partial\phi}{\partial u}X_2\right)\frac{\partial\lambda}{\partial u},\\
\frac{\partial\bar{x}_1}{\partial v} &= [e^{-\theta}\lambda T - e^{\theta}(1-\lambda w)]X_1 - e^{-2\theta}\frac{\lambda}{m}\frac{\partial\phi}{\partial u}X + \frac{e^{-\theta}}{m}\left(\frac{\partial\phi}{\partial v}X_1 - \frac{\partial\phi}{\partial u}X_2\right)\frac{\partial\lambda}{\partial v}.
\end{aligned} \tag{18}$$

We denote by $\bar{S}_1$ the second focal sheet of the above congruence and proceed to find the value of $\lambda$, in order that the coördinates of $\bar{S}_1$ be given by (17). Also we denote by $X'$, $Y'$, $Z'$ the direction-cosines of the normal to $\bar{S}_1$. They are of the form

$$X' = aX + b\left(\frac{\partial\phi}{\partial u}X_1 + \frac{\partial\phi}{\partial v}X_2\right), \tag{19}$$

where $a$ and $b$ are functions of $u$ and $v$ such that

$$\Sigma X'\frac{\partial\bar{x}_1}{\partial u} = 0, \qquad \Sigma X'\frac{\partial x_1}{\partial v} = 0.$$

These equations reduce to

$$\begin{aligned}
e^{-2\theta}\frac{\lambda}{m}\frac{\partial\phi}{\partial v}a - [e^{-\theta}\lambda T + e^{\theta}(1-\lambda w)]\frac{\partial\phi}{\partial v}b = 0,\\
e^{-2\theta}\frac{\lambda}{m}\frac{\partial\phi}{\partial u}a - [e^{-\theta}\lambda T - e^{\theta}(1-\lambda w)]\frac{\partial\phi}{\partial u}b = 0.
\end{aligned} \tag{20}$$

When $S$ is not a surface of revolution, that is when $\theta$ is a function of both $u$ and $v$, the function $\phi$ also is a function of both $u$ and $v$.*

It will be found later that for the cases to be considered the function $a$ is always different from zero.

Eliminating $a$ and $b$ from equations (20), we get $\lambda = 1/w$, so that the coördinates of the second focal sheet are of the form

$$\bar{x}_1 = \bar{x} + \frac{e^{-\theta}}{mw}\left(\frac{\partial \phi}{\partial v} X_1 - \frac{\partial \phi}{\partial u} X_2\right). \tag{21}$$

Now equations (18) become

$$\begin{aligned} \frac{\partial \bar{x}_1}{\partial u} &= -\frac{e^{-\theta}}{mw^2}\frac{\partial \phi}{\partial v}\frac{\partial w}{\partial u} X_1 + e^{-\theta}\left(\frac{T}{w} + \frac{1}{mw^2}\frac{\partial \phi}{\partial u}\frac{\partial w}{\partial u}\right) X_2 - e^{-2\theta}\frac{1}{mw}\frac{\partial \phi}{\partial v} X, \\ \frac{\partial \bar{x}_1}{\partial v} &= e^{-\theta}\left(\frac{T}{w} - \frac{1}{mw^2}\frac{\partial \phi}{\partial v}\frac{\partial w}{\partial v}\right) X_1 + \frac{e^{-\theta}}{mw^2}\frac{\partial \phi}{\partial u}\frac{\partial w}{\partial v} X_2 - e^{-2\theta}\frac{1}{mw}\frac{\partial \phi}{\partial u} X. \end{aligned} \tag{22}$$

Since the congruence is a $W$-congruence, the parametric lines on $\bar{S}_1$ are asymptotic. Hence the necessary and sufficient condition that $\bar{S}_1$ be a minimal surface is that these lines be orthogonal. We shall limit our discussion to $W$-congruences of this kind.†

From (22) it follows that the condition of orthogonality reduces to

$$\frac{\partial w}{\partial u}\frac{\partial w}{\partial v}\left[\left(\frac{\partial \phi}{\partial u}\right)^2 + \left(\frac{\partial \phi}{\partial v}\right)^2\right] + wmT\left[\frac{\partial \phi}{\partial u}\frac{\partial w}{\partial v} - \frac{\partial \phi}{\partial v}\frac{\partial w}{\partial u}\right] + e^{-2\theta}w^2\frac{\partial \phi}{\partial u}\frac{\partial \phi}{\partial v} = 0.$$

Since $w$ is a function of both $u$ and $v$,‡ we can replace this equation by

$$e^{-2\theta}\left[\left(\frac{\partial \phi}{\partial u}\right)^2 + \left(\frac{\partial \phi}{\partial v}\right)^2\right] + mwT\begin{vmatrix} \frac{\partial \phi}{\partial u} & \frac{\partial \phi}{\partial v} \\ \frac{\partial \psi}{\partial u} & \frac{\partial \psi}{\partial v} \end{vmatrix} - w^2\frac{\partial \phi}{\partial u}\frac{\partial \phi}{\partial v}\Big/\frac{\partial \psi}{\partial u}\frac{\partial \psi}{\partial v} = 0,$$

where $\psi$ is given by (13).

If we define a function $\bar{w}$ by

$$\phi = \psi + \bar{w}$$

* For, if $\partial\phi/\partial u = 0$, it follows from (15) that $\Sigma x_0 X_1 = 0$. Differentiate with respect to $v$; this gives $\Sigma x_0 X_2 = 0$, since $\partial\theta/\partial u \neq 0$. Hence $x_0/X = y_0/Y = z_0/Z = \rho$, where $\rho$ is a factor of proportionality. If $x_0$, $y_0$ and $z_0$ be replaced by these values in equations (10) and similar equations in $y_0$ and $z_0$, we are brought to a set of inconsistent equations. Hence $\phi$ is a function of both $u$ and $v$.

† From (19) it is seen that when $a = 0$ the normals to $\bar{S}$ and $\bar{S}_1$ are perpendicular, so that the congruence is normal. But the condition that $\bar{S}_1$ be minimal as well as $\bar{S}$ carries with it the correspondence of the lines of curvature on the two surfaces. However, the lines of curvature correspond on the two focal sheets of a normal $W$-congruence only when these sheets are pseudospherical (BIANCHI, I, p. 283; German translation, p. 244). Hence $a \neq 0$.

‡ As we have excluded the case where $\theta$ is a function of $u$ or $v$ alone, it is seen from (9) that $w$ is a constant if it is not a function of both $u$ and $v$.

and substitute this expression in the above equation, it becomes

$$e^{-2\theta}\left[\left(\frac{\partial\phi}{\partial u}\right)^2+\left(\frac{\partial\phi}{\partial v}\right)^2\right]+2mwT-w^2+mwT\begin{pmatrix}\frac{\partial\bar{w}}{\partial u} & \frac{\partial\bar{w}}{\partial v}\\ \frac{\partial\psi}{\partial u} & \frac{\partial\psi}{\partial v}\end{pmatrix}$$
$$-w^2\begin{pmatrix}\frac{\partial\bar{w}}{\partial u} & \frac{\partial\bar{w}}{\partial v} & \frac{\partial\bar{w}}{\partial u}\frac{\partial\bar{w}}{\partial v}\\ \frac{\partial\psi}{\partial u} & \frac{\partial\psi}{\partial v} & \frac{\partial\psi}{\partial u}\frac{\partial\psi}{\partial v}\end{pmatrix}=0.$$

If this equation be differentiated with respect to $u$ and $v$ and we make use of equations (13) and (16), we find that in each case the resulting equation can be made to vanish identically by taking $\bar{w}$ constant. Now the above equation of condition reduces to

$$e^{-2\theta}\left[\left(\frac{\partial\phi}{\partial u}\right)^2+\left(\frac{\partial\phi}{\partial v}\right)^2\right]+2mwT-w^2=0, \tag{23}$$

and from (13) it follows that

$$\frac{\partial\phi}{\partial u}=e^{2\theta}\frac{\partial w}{\partial u},\qquad \frac{\partial\phi}{\partial v}=-e^{2\theta}\frac{\partial w}{\partial v}. \tag{A}$$

In consequence of these equations and (15), equations (14) can be written in the form

$$m\frac{\partial T}{\partial u}=\frac{\partial w}{\partial u}-m\frac{\partial\phi}{\partial u},\qquad m\frac{\partial T}{\partial v}=\frac{\partial w}{\partial v}-m\frac{\partial\phi}{\partial v}.$$

Thus far the function $\phi$ has been defined only to within an additive constant, hence in all generality we can write the integral of the above equations in the form

$$mT=w-m\phi. \tag{24}$$

Now equations (16) and (23) assume the fundamental forms

$$\frac{\partial^2\phi}{\partial u^2}=\frac{\partial\theta}{\partial u}\frac{\partial\phi}{\partial u}-\frac{\partial\theta}{\partial v}\frac{\partial\phi}{\partial v}+me^{2\theta}w+m\phi-w,$$

$$\frac{\partial^2\phi}{\partial u\partial v}=\frac{\partial\theta}{\partial v}\frac{\partial\phi}{\partial u}+\frac{\partial\theta}{\partial u}\frac{\partial\phi}{\partial v}, \tag{B}$$

$$\frac{\partial^2\phi}{\partial v^2}=-\frac{\partial\theta}{\partial u}\frac{\partial\phi}{\partial u}+\frac{\partial\theta}{\partial v}\frac{\partial\phi}{\partial v}+me^{2\theta}w-m\phi+w,$$

$$e^{-2\theta}\left[\left(\frac{\partial\phi}{\partial u}\right)^2+\left(\frac{\partial\phi}{\partial v}\right)^2\right]=2m\phi w-w^2. \tag{C}$$

Moreover equations (20) reduce to

$$a=e^{\theta}mTb.$$

Substituting in (19) and expressing the condition

$$X'^2 + Y'^2 + Z'^2 = 1,$$

we get

$$b^2 e^{2\theta} m^2 \phi^2 = 1.$$

From (18) it is found that we must take

$$b = -e^{-\theta}/m\phi,$$

so that (19) becomes

$$X' = \left(1 - \frac{w}{m\phi}\right) X - \frac{e^{-\theta}}{m\phi}\left(\frac{\partial \phi}{\partial u} X_1 + \frac{\partial \phi}{\partial v} X_2\right). \tag{25}$$

From (22) we find for the linear element of $\bar{S}_1$

$$d\bar{s}_1^2 = e^{-2\theta}\frac{\phi^2}{w^2}(du^2 + dv^2); \tag{26}$$

and consequently the linear element of its spherical representation is

$$ds_1'^2 = e^{2\theta}\frac{w^2}{\phi^2}(du^2 + dv^2). \tag{27}$$

The transformation (21) from the surface $S$ to the second minimal surface $\bar{S}_1$ was discovered by THYBAUT.* By retracing the steps in the foregoing development it is readily shown that *every pair of functions $\phi$, $w$ satisfying the fundamental equations (A), (B), (C), determine a transformation of Thybaut.*

Denote by $S_1$ the adjoint minimal surface of $\bar{S}_1$; its coördinates are given by

$$\frac{\partial x_1}{\partial u} = -\frac{\partial \bar{x}_1}{\partial v} = e^{-\theta}\frac{\phi}{w} X_1', \qquad \frac{\partial x_1}{\partial v} = \frac{\partial \bar{x}_1}{\partial u} = e^{-\theta}\frac{\phi}{w} X_2', \tag{28}$$

where

$$\begin{aligned} X_1' &= \left[\frac{e^{-2\theta}}{m\phi w}\left(\frac{\partial \phi}{\partial u}\right)^2 - 1\right] X_1 + \frac{e^{-2\theta}}{m\phi w}\frac{\partial \phi}{\partial u}\frac{\partial \phi}{\partial v} X_2 + \frac{e^{-\theta}}{m\phi}\frac{\partial \phi}{\partial u} X, \\ X_2' &= -\frac{e^{-2\theta}}{m\phi w}\frac{\partial \phi}{\partial u}\frac{\partial \phi}{\partial v} X_1 + \left[1 - \frac{e^{-2\theta}}{m\phi w}\left(\frac{\partial \phi}{\partial v}\right)^2\right] X_2 - \frac{e^{-\theta}}{m\phi}\frac{\partial \phi}{\partial v} X. \end{aligned} \tag{29}$$

As thus defined the functions $X_1'$, $Y_1'$, $Z_1'$; $X_2'$, $Y_2'$, $Z_2'$, are the direction-cosines of the tangents to the curves $v = \text{const.}$, $u = \text{const.}$, respectively, on $S_1$.

In consequence of equations (7) we can write the integral of equations (28) in the form

$$x_1 = x - \frac{1}{m}\left[\frac{e^{-\theta}}{w}\frac{\partial \phi}{\partial u} X_1 + \frac{e^{-\theta}}{w}\frac{\partial \phi}{\partial v} X_2 + X\right]. \tag{30}$$

This equation and similar ones for $y_1$ and $z_1$ define the transformation from

* *Sur la déformation du paraboloïde et sur quelques problèmes qui s'y rattachent.* Annales de l'École Normale, ser. 3, vol. 14 (1897), p. 45.

$S$ to $S_1$ which is entirely complementary to the THYBAUT transformation of their adjoint surfaces $\bar{S}$ and $\bar{S}_1$.

Referring to equations (25), we see that equation (30) can be written in the form

$$x_1 - \frac{\phi}{w} X' = x - \frac{\phi}{w} X. \tag{30'}$$

Hence the normals to $S$ and $S_1$ at corresponding points $M$, $M_1$, meet in a point $M_0$ which is distant $\phi/w$ from $M$ and $M_1$. Therefore the surfaces $S$ and $S_1$ are the sheets of the envelope of the spheres of radius $\phi/w$ whose centers are on the surface which is the locus of $M_0$. It can be shown that the latter surface is applicable to a paraboloid of revolution.*

## § 2. *Inverse transformation of Thybaut.*

Since $\bar{S}$ and $\bar{S}_1$ are the focal sheets of a congruence the relation between them is entirely reciprocal so that there is a THYBAUT transformation from the latter into the former. We denote the transformation functions by $\phi_1$, $w_1$, $m_1$ and seek their form.

The equations of transformation are of the form

$$\bar{x} = \bar{x}_1 + \frac{e^{-\theta_1}}{m_1 w_1}\left(\frac{\partial \phi_1}{\partial v} X_1' - \frac{\partial \phi_1}{\partial u} X_2'\right),$$

where, in consequence of (27),

$$e^{-\theta_1} = e^{\theta} \frac{w}{\phi}.$$

Replacing $\bar{x}_1$, $X_1'$, $X_2'$ by their values from (21) and (29), we get an equation of the form

$$AX_1 + BX_2 + CX = 0,$$

where $A$, $B$, $C$ are determinate functions of the above quantities. Since the same relation holds for $Y$ and $Z$, these functions must vanish identically. This gives the three equations

$$\frac{1}{mw}\frac{\partial \phi}{\partial v} + \frac{1}{m_1 w_1 \phi}\left[\frac{1}{m\phi}\left(\frac{\partial \phi}{\partial u}\right)^2 - e^{2\theta} w\right]\frac{\partial \phi_1}{\partial v} + \frac{1}{m m_1 w_1 \phi^2}\frac{\partial \phi}{\partial u}\frac{\partial \phi}{\partial v}\frac{\partial \phi_1}{\partial u} = 0,$$

$$-\frac{1}{mw}\frac{\partial \phi}{\partial u} + \frac{1}{m m_1 w_1 \phi^2}\frac{\partial \phi}{\partial u}\frac{\partial \phi}{\partial v}\frac{\partial \phi_1}{\partial v} + \frac{1}{m_1 w_1 \phi}\left[\frac{1}{m\phi}\left(\frac{\partial \phi}{\partial v}\right)^2 - e^{2\theta} w\right]\frac{\partial \phi_1}{\partial u} = 0,$$

$$\frac{\partial \phi}{\partial u}\frac{\partial \phi_1}{\partial v} + \frac{\partial \phi}{\partial v}\frac{\partial \phi_1}{\partial u} = 0.$$

We replace the last equation by

$$\frac{\partial \phi_1}{\partial u} = -\rho \frac{\partial \phi}{\partial u}, \qquad \frac{\partial \phi_1}{\partial v} = \rho \frac{\partial \phi}{\partial v},$$

* BIANCHI, II, p. 333.

and find by substitution that the first two equations are satisfied if

$$\rho = \frac{\phi m_1 w_1}{m w^2} e^{-2\theta}.$$

Hence

$$\frac{\partial \phi_1}{\partial u} = -\frac{m_1 w_1 \phi}{m w^2} e^{-2\theta} \frac{\partial \phi}{\partial u}, \qquad \frac{\partial \phi_1}{\partial v} = \frac{m_1 w_1 \phi}{m w^2} e^{-2\theta} \frac{\partial \phi}{\partial v}. \tag{31}$$

But $\phi_1$ and $w_1$ must satisfy equations of the form $(A)$, which are for the present case

$$\frac{\partial \phi_1}{\partial u} = e^{-2\theta} \frac{\phi^2}{w^2} \frac{\partial w_1}{\partial u}, \qquad \frac{\partial \phi_1}{\partial v} = -e^{-2\theta} \frac{\phi^2}{w^2} \frac{\partial w_1}{\partial v}. \tag{32}$$

Subtracting these equations from (31), we get

$$\frac{\partial w_1}{\partial u} = -\frac{m_1 w_1}{m \phi} \frac{\partial \phi}{\partial u}, \qquad \frac{\partial w_1}{\partial v} = -\frac{m_1 w_1}{m \phi} \frac{\partial \phi}{\partial v},$$

of which the integral is

$$w_1 = \left(\frac{1}{\phi}\right)^{\frac{m_1}{m}},$$

with a particular choice of the constant of integration which has no effect upon the generality of the solution.

Now equations (31) reduce to

$$\frac{\partial \phi_1}{\partial u} = -\frac{m_1}{m w^2} \left(\frac{1}{\phi}\right)^{\frac{m_1}{m}-1} e^{-2\theta} \frac{\partial \phi}{\partial u}, \qquad \frac{\partial \phi_1}{\partial v} = \frac{m_1}{m w^2} \left(\frac{1}{\phi}\right)^{\frac{m_1}{m}-1} e^{-2\theta} \frac{\partial \phi}{\partial v}. \tag{33}$$

The functions $\phi_1$ and $w_1$ must satisfy equations of the form $(B)$. The second of these is

$$\frac{\partial^2 \phi_1}{\partial u \partial v} = \frac{\partial}{\partial v} \log\left(e^{-\theta} \frac{\phi}{w}\right) \frac{\partial \phi_1}{\partial u} + \frac{\partial}{\partial u} \log\left(e^{-\theta} \frac{\phi}{w}\right) \frac{\partial \phi_1}{\partial v}.$$

When the values from (33) are substituted in this equation, it reduces to $m_1 = m$. Consequently by means of $(A)$ equations (33) can be reduced to

$$\frac{\partial \phi_1}{\partial u} = -\frac{1}{w^2} \frac{\partial w}{\partial u}, \qquad \frac{\partial \phi_1}{\partial v} = -\frac{1}{w^2} \frac{\partial w}{\partial v}.$$

Hence the functions determining the transformation from $\bar{S}_1$ into $\bar{S}$ are given by

$$\phi_1 = \frac{1}{w}, \qquad w_1 = \frac{1}{\phi}, \qquad m_1 = m. \tag{34}$$

## § 3. *The associate surfaces* $\Sigma_0$.

Equation (9) is the tangential equation of the surfaces with the same spherical representation of their lines of curvature as $S$. Each solution $W$ of this

equation determines such a surface. We call them *surfaces* $\Sigma$. The function $W$ gives the distance from the origin to the tangent plane and the rectangular coördinates $(\xi, \eta, \zeta)$ are of the form *

$$\xi = WX + e^{2\theta}\left(\frac{\partial W}{\partial u}\frac{\partial X}{\partial u} + \frac{\partial W}{\partial v}\frac{\partial X}{\partial v}\right),$$

or in consequence of (6),

$$\xi = WX + e^{\theta}\left(\frac{\partial W}{\partial u}X_1 - \frac{\partial W}{\partial v}X_2\right). \tag{35}$$

These surfaces $\Sigma$ are evidently associate to $\bar{S}$, for the lines of curvature of the former have the same representation as the asymptotic lines of the latter.

Since the function $w$ of a THYBAUT transformation of $\bar{S}$ is a solution of equation (9), every transformation of this kind carries with it the determination of a surface associate to $\bar{S}$. We denote this surface by $\Sigma_0$ and by $2q$ the square of the distance from a point on $\Sigma_0$ to the origin. From (35), $(A)$ and $(C)$ we get

$$2q = 2m\phi w.$$

Denoting by $w_{rs}$ the second derived covariants of the function $w$ with respect to the linear element of the spherical representation of $S$ we get from $(A)$ and $(B)$

$$w_{11} = \frac{\partial^2 w}{\partial u^2} + \frac{\partial \theta}{\partial u}\frac{\partial w}{\partial u} - \frac{\partial \theta}{\partial v}\frac{\partial w}{\partial v} = mw + e^{-2\theta}(m\phi - w),$$

$$w_{22} = \frac{\partial^2 w}{\partial v^2} - \frac{\partial \theta}{\partial u}\frac{\partial w}{\partial u} + \frac{\partial \theta}{\partial v}\frac{\partial w}{\partial v} = -mw + e^{-2\theta}(m\phi - w).$$

The sum of the radii of principal curvature of $\Sigma_0$ has the value †

$$\rho_1 + \rho_2 = e^{2\theta}(w_{11} + w_{22}) + 2w = 2m\phi,$$

so that the above equation can be written

$$2q = (\rho_1 + \rho_2)w. \tag{36}$$

## § 4. *General transformations of surfaces* $\Sigma$.

MOUTARD has established the following theorem: ‡ Given two solutions $\psi_1$, $\psi_2$, of an equation of the form

$$\frac{\partial^2 \psi}{\partial u \partial v} = M\psi; \tag{37}$$

the equations

$$\frac{\partial}{\partial u}(\psi_1\psi_1') = \psi_1^2\frac{\partial}{\partial u}\left(\frac{\psi_2}{\psi_1}\right), \qquad \frac{\partial}{\partial v}(\psi_1\psi_1') = -\psi_1^2\frac{\partial}{\partial v}\left(\frac{\psi_2}{\psi_1}\right), \tag{38}$$

* BIANCHI, *Lezioni*, I, p. 173; German translation, p. 141.
† BIANCHI, *Lezioni*, I, p. 173; German translation, p. 141.
‡ Ibid., II, p. 69.

are consistent, and the function $\psi_1'$ thus defined is such that

$$\frac{\partial^2 \psi_1'}{\partial u \partial v} = \psi_1 \frac{\partial^2}{\partial u \partial v}\left(\frac{1}{\psi_1}\right)\psi_1'. \tag{39}$$

By means of this theorem we shall establish a transformation of a surface $\Sigma$ into a surface $\Sigma_1$ with the same spherical representation of its lines of curvature as a transform $S_1$ of $S$.

If we put

$$w = e^{-\theta}\psi, \tag{40}$$

equation (9) becomes

$$\frac{\partial^2 \psi}{\partial u \partial v} = e^{-\theta}\frac{\partial^2 e^{\theta}}{\partial u \partial v}\psi, \tag{41}$$

which evidently is of the form (37).

Suppose now that we have given a surface $\Sigma$ determined by a solution $W$ of equation (9) and a pair of functions $\phi, w$, giving a THYBAUT transformation of $S$. Hence two solutions of (41) are

$$\psi_1 = e^{\theta} w, \qquad \psi_2 = e^{\theta} W.$$

Equations (38) reduce to

$$\frac{\partial}{\partial u}(e^{\theta} w \psi_1') = e^{2\theta} w^2 \frac{\partial}{\partial u}\left(\frac{W}{w}\right), \qquad \frac{\partial}{\partial v}(e^{\theta} w \psi_1') = -e^{2\theta} w^2 \frac{\partial}{\partial v}\left(\frac{W}{w}\right) \tag{42}$$

and the corresponding equation (39) is

$$\frac{\partial^2 \psi_1'}{\partial u \partial v} = e^{\theta} w \frac{\partial^2}{\partial u \partial v}\left(\frac{e^{-\theta}}{w}\right)\psi_1'. \tag{43}$$

The tangential equation of the surfaces $\Sigma_1$ is evidently

$$\frac{\partial^2 W_1}{\partial u \partial v} - \frac{\partial}{\partial v}\log\left(e^{\theta}\frac{w}{\phi}\right)\frac{\partial W_1}{\partial u} - \frac{\partial}{\partial u}\log\left(e^{\theta}\frac{w}{\phi}\right)\frac{\partial W_1}{\partial v} = 0. \tag{44}$$

If we put

$$W_1 = -\frac{e^{\theta} w}{\phi}\psi_1',$$

the equation for $\psi_1'$ reduces to (43), in consequence of equations $(A)$ and $(C)$. Hence the function $W_1$, determined by the quadratures

$$\frac{\partial}{\partial u}(\phi W_1) = -e^{2\theta} w^2 \frac{\partial}{\partial u}\left(\frac{W}{w}\right), \qquad \frac{\partial}{\partial v}(\phi W_1) = e^{2\theta} w^2 \frac{\partial}{\partial v}\left(\frac{W}{w}\right), \tag{45}$$

is a solution of equation (44).

The surface whose coördinates are of the form

$$\xi_1 = W_1 X' + e^{-\theta}\frac{\phi}{w}\left(\frac{\partial W_1}{\partial u}X_1' - \frac{\partial W_1}{\partial v}X_2'\right) \tag{46}$$

is therefore associate to the transform $\bar{S}_1$ of $\bar{S}$ by means of the functions $\phi$, $w$. If the expressions for $X'$, $X_1'$, $X_2'$, given by (25) and (29) and of $\partial W_1/\partial u$, $\partial W_1/\partial v$, be substituted in (46), it can be reduced to the form

$$\xi_1 = \xi + (X' - X)R, \tag{47}$$

where we have put

$$R = \frac{1}{w}\left[ m\phi W_1 + \left( \frac{\partial W}{\partial u}\frac{\partial \phi}{\partial u} - \frac{\partial W}{\partial v}\frac{\partial \phi}{\partial v} \right) - W(m\phi - w) \right]. \tag{48}$$

From (47) it is evident that the surfaces $\Sigma$ and $\Sigma_1$ are the sheets of the envelopes of spheres of radius $R$ whose centers lie on the surface $\bar{\Sigma}$ defined by

$$\bar{\xi} = \xi - RX, \qquad \bar{\eta} = \eta - RY, \qquad \bar{\zeta} = \zeta - RZ. \tag{49}$$

From these expressions we get by differentiation

$$\frac{\partial \bar{\xi}}{\partial u} = Ae^{\theta}X_1 - \frac{\partial R}{\partial u}X, \qquad \frac{\partial \bar{\xi}}{\partial v} = -Be^{\theta}X_2 - \frac{\partial R}{\partial v}X,$$

where we have put for the sake of brevity

$$\begin{aligned} A &= \frac{\partial^2 W}{\partial u^2} + \frac{\partial \theta}{\partial u}\frac{\partial W}{\partial u} - \frac{\partial \theta}{\partial v}\frac{\partial W}{\partial v} + (W - R)e^{-2\theta}, \\ B &= \frac{\partial^2 W}{\partial v^2} - \frac{\partial \theta}{\partial u}\frac{\partial W}{\partial u} + \frac{\partial \theta}{\partial v}\frac{\partial W}{\partial v} + (W - R)e^{-2\theta}. \end{aligned} \tag{49'}$$

But it is readily found from (48) that

$$\frac{\partial R}{\partial u} = \frac{A}{w}\frac{\partial \phi}{\partial u}, \qquad \frac{\partial R}{\partial v} = -\frac{B}{w}\frac{\partial \phi}{\partial v},$$

so that the above equations can be written

$$\frac{\partial \bar{\xi}}{\partial u} = A\left( e^{\theta}X_1 - \frac{1}{w}\frac{\partial \phi}{\partial u}X \right), \qquad \frac{\partial \bar{\xi}}{\partial v} = B\left( -e^{\theta}X_2 + \frac{1}{w}\frac{\partial \phi}{\partial v}X \right). \tag{50}$$

From these expressions and similar ones for $\bar{\eta}$ and $\bar{\zeta}$ we find that the direction-cosines of the normal to the surface $\bar{\Sigma}$ are of the form

$$\bar{X} = \sqrt{\frac{w}{2m\phi}}\left[ X + \frac{e^{-\theta}}{w}\left( \frac{\partial \phi}{\partial u}X_1 + \frac{\partial \phi}{\partial v}X_2 \right) \right]. \tag{51}$$

As defined, a surface $\bar{\Sigma}$ is the locus of the centers of spheres whose envelope consists of a surface $\Sigma$ and its transform by means of a set of functions $m$, $w$, $\phi$. Since the expression (51) for $\bar{X}$ involves only the latter functions, all the transforms, by means of these functions, of the surfaces $\Sigma$ correspond with parallelism of tangent planes. And furthermore it follows from (50) that the tangents to the parametric curves on these surfaces $\bar{\Sigma}$ are parallel; hence the parametric curves form a conjugate system.

§ 5. *Special cases.*

We have seen that the minimal surfaces $S$ and $S_1$ are the sheets of the envelope of a family of spheres. Now we shall show that this transformation is the same as that just given for any surface $\Sigma$.

From (30) and (25) we get

$$(52) \quad W_1 = \Sigma x_1 X' = \left(1 - \frac{w}{m\phi}\right) W - \frac{e^{-\theta}}{m\phi}\left[\Sigma x X_1 \frac{\partial \phi}{\partial u} + \Sigma x X_2 \frac{\partial \phi}{\partial v}\right] + \frac{1}{m}.$$

If we differentiate the equation

$$W = \Sigma x X,$$

we get

$$\frac{\partial W}{\partial u} = e^{-\theta}\Sigma x X_1, \qquad \frac{\partial W}{\partial v} = -e^{-\theta}\Sigma x X_2.$$

When these values are substituted in (52) and (48), we have

$$R = \frac{\phi}{w},$$

in which case equations (47) reduce to (30′). Hence $S_1$ is the transform of $S$ in the sense of the generalized transformation of any surface $\Sigma$.

Let us now apply this transformation to the surface $\Sigma_0$, that is, we have $W = w$. From (45) it is seen that $W_1\phi$ is constant. Taking $W_1 = 1/\phi$ we get, as we have seen before, the surface $\Sigma_{10}$ which determines the inverse THYBAUT transformation from $\bar{S}_1$ into $\bar{S}$. Now from (48) we have

$$R = m\left(\frac{1}{w} + \phi\right).$$

From (34) it is seen that this expression may be written

$$R = m\left(\phi_1 + \frac{1}{w_1}\right),$$

which puts in evidence the reciprocal character of the transformation.

In consequence of equations $(A)$ equations (45) can be written in the form

$$(45') \quad \begin{aligned} \phi \frac{\partial W_1}{\partial u} &= -e^{2\theta} w \frac{\partial W}{\partial u} + (W - W_1)\frac{\partial \phi}{\partial u}, \\ \phi \frac{\partial W_1}{\partial v} &= e^{2\theta} w \frac{\partial W}{\partial v} + (W - W_1)\frac{\partial \phi}{\partial v}. \end{aligned}$$

From this it is seen that if $W_1$ and $W$ are any functions satisfying this equation, so also are $W_1 + a$ and $W + a$ for all values of the constant $a$. Hence *surfaces parallel to a given surface* $\Sigma$ *are transformed into surfaces parallel to* $\Sigma_1$ *and at the same distance.*

Suppose we have two solutions $W$ and $W'$ of equation (9) and that the corresponding solutions of (45′) are denoted by $W_1$ and $W_1'$. It is evident that $aW + bW'$ is a solution of equation (9) for all values of the constants $a$, $b$, and the corresponding solution of (45′) is $aW_1 + bW_1'$. Hence *the locus of a point dividing in constant ratio the join of corresponding points on two surfaces* $\Sigma$, $\Sigma'$ *is a surface of the same kind. And the transform of this locus by means of the functions* $m$, $\phi$, $w$, *divides in the same ratio the join of corresponding points of the transforms of* $\Sigma$ *and* $\Sigma'$ *by means of the same functions.*

When in particular $W'$ is $w$, then $W_1$ is $1/\phi$ as we have seen. From this we remark that by varying the constant of integration obtained from the quadratures (45), we get the surfaces which divide in constant ratio the joins of corresponding points on the surface for which the constant is zero, and the surface $\Sigma_{10}$ previously defined.

## § 6. *Theorem of permutability.*

There is a theorem of permutability for the transformations of THYBAUT very similar to the theorem of permutability of the BÄCKLUND transformations, as discovered by BIANCHI. It is as follows: Given two transforms $\bar{S}_1$ and $\bar{S}_2$ of a minimal surface $\bar{S}$ by means of transformations involving respective constants $m_1$, $m_2$; there exists a minimal surface $\bar{S}'$ which is the THYBAUT transform of $\bar{S}_1$ and $\bar{S}_2$ by means of transformations involving the same constants in inverse order.

This theorem is a direct result of a more general theorem due to BIANCHI.* Instead of considering the surfaces $\bar{S}$, $\bar{S}_1$, $\bar{S}_2$, we take their adjoints $S$, $S_1$, $S_2$. We have seen that $S$ and $S_1$ are sheets of an envelope of spheres whose centers lie on a surface; and the same is true of $S$ and $S_2$. DARBOUX has considered pairs of surfaces which are the sheets of an envelope of spheres in such a way that the two sheets are represented conformally upon one another; he has found that both the surfaces are isothermic and their lines of curvature correspond. BIANCHI * has made a profound study of these transformations of one isothermic surface into another and has denoted such a transformation by $D_m$, thus putting in evidence the constant $m$ which appears in the equations. He has established a theorem of permutability of these transformations which leads to the theorem stated above. We shall write down the results of his investigation for the case where the given isothermal surface is minimal and refer the reader to his memoir for their derivation.

The surfaces $S_1$ and $S_2$ are defined by equations of the form (30), when the functions $m$, $\phi$, $w$ are replaced by $m_1$, $\phi_1$, $w_1$ and $m_2$, $\phi_2$, $w_2$ successively. In like manner transforms $S'$ and $S''$ of $S_1$ and $S_2$ respectively are defined by

* *Ricerche sulle superficie isoterme*, etc., loc. cit.

equations of the form

$$x' = x_1 - \frac{1}{m_2}\left[\frac{e^{-\theta_1}}{w_1'}\frac{\partial \phi_1'}{\partial u}X_1^{(1)} + \frac{e^{-\theta_1}}{w_1'}\frac{\partial \phi_1'}{\partial v}X_2^{(1)} + X^{(1)}\right],$$

$$x'' = x_2 - \frac{1}{m_1}\left[\frac{e^{-\theta_2}}{w_2'}\frac{\partial \phi_2'}{\partial u}X_1^{(2)} + \frac{e^{-\theta_2}}{w_2'}\frac{\partial \phi_2'}{\partial v}X_2^{(2)} + X^{(2)}\right].$$

From (27) it follows that

$$e^{-\theta_1} = e^{\theta}\frac{w_1}{\phi_1}, \qquad e^{-\theta_2} = e^{\theta}\frac{w_2}{\phi_2}; \tag{53}$$

and $X_1^{(1)}$, $X_2^{(1)}$, $X^{(1)}$; $X_1^{(2)}$, $X_2^{(2)}$, $X^{(2)}$; are given by (25) and (29) when $\phi$, $w$, $m$ are replaced by $\phi_1$, $w_1$, $m_1$ and $\phi_2$, $w_2$, $m_2$ respectively.

BIANCHI shows* that $S'$ and $S''$ coincide when

$$\begin{aligned} w_1' &= \frac{\Phi_1}{\phi_1} + (m_2 - m_1)w_2, \qquad & \phi_1' &= \frac{\Phi_1}{w_1} + (m_2 - m_1)\phi_2, \\ w_2' &= \frac{\Phi_2}{\phi_2} + (m_1 - m_2)w_1, \qquad & \phi_2' &= \frac{\Phi_2}{w_2} + (m_1 - m_2)\phi_1, \end{aligned} \tag{54}$$

where

$$\Phi_1 = e^{-2\theta}\left(\frac{\partial \phi_1}{\partial u}\frac{\partial \phi_2}{\partial u} + \frac{\partial \phi_1}{\partial v}\frac{\partial \phi_2}{\partial v}\right) + w_1 w_2 - m_2(w_1\phi_2 + w_2\phi_1),$$

$$\Phi_2 = e^{-2\theta}\left(\frac{\partial \phi_1}{\partial u}\frac{\partial \phi_2}{\partial u} + \frac{\partial \phi_1}{\partial v}\frac{\partial \phi_2}{\partial v}\right) + w_1 w_2 - m_1(w_1\phi_2 + w_2\phi_1). \tag{55}$$

Moreover, the above functions satisfy fundamental equations for $S_1$ and $S_2$ similar to $(A)$, $(B)$, $(C)$ for $S$.

From (21) it follows that the THYBAUT transform of $\bar{S}_1$ by means of the above functions is defined by equations of the form

$$\bar{x}' = \bar{x}_1 + \frac{e^{-\theta_1}}{m_2 w_1'}\left(\frac{\partial \phi_1'}{\partial v}X_1^{(1)} - \frac{\partial \phi_1'}{\partial u}X_2^{(1)}\right).$$

By means of (53) we get

$$\begin{aligned} e^{-\theta_1}\frac{\partial \phi_1'}{\partial u} &= -\frac{\Phi_1}{\phi_1 w_1}e^{-\theta}\frac{\partial \phi_1}{\partial u} - (m_2 - m_1)e^{-\theta}\frac{\partial \phi_2}{\partial u}, \\ e^{-\theta_1}\frac{\partial \phi_1'}{\partial v} &= \frac{\Phi_1}{\phi_1 w_1}e^{-\theta}\frac{\partial \phi_1}{\partial v} + (m_2 - m_1)e^{-\theta}\frac{\partial \phi_2}{\partial v}. \end{aligned} \tag{56}$$

* Loc. cit., p. 119. It must be remarked that BIANCHI has put $\lambda = e^{-\theta}(\partial\phi/\partial u)$, $\mu = e^{-\theta}(\partial\phi/\partial v)$ to reduce his formulæ to the form, p. 119, and that his function $\sigma$ reduces to $w$ when $S$ is minimal.

Hence the above equation can be reduced to

$$\begin{aligned}\bar{x}' = \bar{x} + \frac{m_2 - m_1}{m_1 m_2 w_1' \phi_1} e^{-\theta} \Bigg\{ & \left[ (m_1\phi_1 - w_1)\frac{\partial \phi_2}{\partial v} - (m_2\phi_2 - w_2)\frac{\partial \phi_1}{\partial v} \right] X_1 \\ & - \left[ (m_1\phi_1 - w_1)\frac{\partial \phi_2}{\partial u} - (m_2\phi_2 - w_2)\frac{\partial \phi_1}{\partial u} \right] X_2 \\ & + e^{-\theta}\left( \frac{\partial \phi_2}{\partial v}\frac{\partial \phi_1}{\partial u} - \frac{\partial \phi_2}{\partial u}\frac{\partial \phi_1}{\partial v} \right) X \Bigg\}. \end{aligned} \tag{57}$$

From (54) and (55) it follows that

$$w_1'\phi_1 = w_2'\phi_2.$$

Hence the right-hand member of equation (57) is symmetrical with respect to the sets of functions $m_1, \phi_1, w_1$; $m_2, \phi_2, w_2$. Consequently $\bar{S}'$ is also the transform of $\bar{S}_2$ by means of the functions $m_1, \phi_2', w_2'$.

It follows from (57) that $\bar{S}$ and $\bar{S}'$ coincide when $m_1 = m_2$ and only in this case.

We shall now show that the theorem of permutability of THYBAUT transformations, as just derived, can be extended to the transformations, which we have found, of surfaces with the same spherical representation of their lines of curvature as $S$.

Given a surface $\Sigma$ and as before denote by $W$ the distance from the origin of its tangent plane. By means of the functions $(m_1, \phi_1, w_1)$ and $(m_2, \phi_2, w_2)$ we transform $\Sigma$ into $\Sigma_1$ and $\Sigma_2$ respectively. Denote by $W_1$ and $W_2$ the distances of the tangent planes of these respective surfaces from the origin; these functions are given by the quadratures (45)

$$\frac{\partial}{\partial u}(\phi_1 W_1) = -e^{2\theta} w_1^2 \frac{\partial}{\partial u}\left(\frac{W}{w_1}\right), \qquad \frac{\partial}{\partial v}(\phi_1 W_1) = e^{2\theta} w_1^2 \frac{\partial}{\partial v}\left(\frac{W}{w_1}\right), \tag{58}$$

$$\frac{\partial}{\partial u}(\phi_2 W_2) = -e^{2\theta} w_2^2 \frac{\partial}{\partial u}\left(\frac{W}{w_2}\right), \qquad \frac{\partial}{\partial v}(\phi_2 W_2) = e^{2\theta} w_2^2 \frac{\partial}{\partial v}\left(\frac{W}{w_2}\right). \tag{59}$$

By means of the functions $(m_2, \phi_1', w_1')$ and $(m_1, \phi_2', w_2')$, given by (54) the surfaces $\Sigma_1$ and $\Sigma_2$ can be transformed into surfaces with the same representation of their lines of curvature as $S'$. We shall show that the two new surfaces coincide.

In order that this may be true the function $W'$, representing the distance from the origin to the tangent plane to $\Sigma'$, must satisfy the four equations

$$\frac{\partial}{\partial u}(\phi_1' W') = -e^{-2\theta}\frac{\phi_1^2}{w_1^2} w_1'^2 \frac{\partial}{\partial u}\left(\frac{W_1}{w_1'}\right), \qquad \frac{\partial}{\partial v}(\phi_1' W') = e^{-2\theta}\frac{\phi_1^2}{w_1^2} w_1'^2 \frac{\partial}{\partial v}\left(\frac{W_1}{w_1'}\right). \tag{60}$$

$$\frac{\partial}{\partial u}(\phi_2' W') = -e^{-2\theta}\frac{\phi_2^2}{w_2^2} w_2'^2 \frac{\partial}{\partial u}\left(\frac{W_2}{w_2'}\right), \qquad \frac{\partial}{\partial v}(\phi_2' W') = e^{-2\theta}\frac{\phi_2^2}{w_2^2} w_2'^2 \frac{\partial}{\partial v}\left(\frac{W_2}{w_2'}\right). \tag{61}$$

Similar to equations $(A)$ we have the set

$$\frac{\partial \phi_i'}{\partial u} = e^{-2\theta}\frac{\phi_i^2}{w_i^2}\frac{\partial w_i'}{\partial u}, \qquad \frac{\partial \phi_i'}{\partial v} = -e^{-2\theta}\frac{\phi_i^2}{w_i^2}\frac{\partial w_i'}{\partial v} \qquad (i=1,2).$$

If we make use of these equations and (56) in the result obtained by eliminating $\partial W'/\partial u$ from the first of equations (60) and (61), we can reduce it to the form

$$(62)\quad \Omega'\left(\frac{1}{w_2}\frac{\partial \phi_2}{\partial u}-\frac{1}{w_1}\frac{\partial \phi_1}{\partial u}\right)[\Omega' W'-\Omega W+(m_2-m_1)(\phi_1 W_1 w_2-\phi_2 W_2 w_1)]=0,$$

where for the sake of brevity we have put

$$(63)\quad \begin{aligned}\Omega' &= \Phi_1+(m_2-m_1)\phi_2 w_1 = \Phi_2+(m_1-m_2)\phi_1 w_2,\\ \Omega &= \Phi_1+(m_2-m_1)\phi_1 w_2 = \Phi_2+(m_1-m_2)\phi_2 w_1.\end{aligned}$$

In like manner the elimination of $\partial W'/\partial v$ from the second of equations (60) and (61) leads to

$$(64)\quad \Omega'\left(\frac{1}{w_2}\frac{\partial \phi_2}{\partial v}-\frac{1}{w_1}\frac{\partial \phi_1}{\partial v}\right)[\Omega' W'-\Omega W+(m_2-m_1)(\phi_1 W_1 w_2-\phi_2 W_2 w_1)]=0.$$

From (63) we get by differentiation

$$\frac{\partial \Omega'}{\partial u} = (m_2-m_1)e^{-2\theta}\left(\phi_2\frac{\partial \phi_1}{\partial u}-\phi_1\frac{\partial \phi_2}{\partial u}\right),$$

$$\frac{\partial \Omega'}{\partial v} = (m_1-m_2)e^{-2\theta}\left(\phi_2\frac{\partial \phi_1}{\partial v}-\phi_1\frac{\partial \phi_2}{\partial v}\right).$$

Hence $\Omega'$ cannot vanish unless $\phi_2$ is a function of $\phi_1$. Moreover, it is only in this case that the quantities in parentheses in (62) and (64) can vanish simultaneously. Hence the expression in brackets in these two equations must vanish, so that

$$(65)\quad \Omega' W' = (m_1-m_2)(\phi_1 W_1 w_2-\phi_2 W_2 w_1)+\Omega W.$$

It is readily shown that this value of $W'$ satisfies equations (60) and (61). Hence we have the following theorem of permutability:

*If a surface $\Sigma$ with isothermal representation of its lines of curvature be transformed into two surfaces $\Sigma_1$ and $\Sigma_2$ of the same kind by means of transformations $T_{m_1}$, $T_{m_2}$ respectively, there exists a surface $\Sigma'$ which is the transform of $\Sigma_1$ and $\Sigma_2$ by transformations $T'_{m_2}$, $T'_{m_1}$ respectively; and this surface can be obtained without quadratures.*

We have seen that the surfaces $\Sigma$ and $\Sigma_1$ form the envelope of a doubly infinite family of spheres whose centers lie on a surface $\bar{\Sigma}_1$ with a conjugate system of lines in correspondence with the lines of curvature on $\Sigma$. In like manner there are surfaces $\bar{\Sigma}_2$, $\bar{\Sigma}_1'$, $\bar{\Sigma}_2'$, which are the loci of the centers of the spheres envel-

oped by the pairs of surfaces $\Sigma, \Sigma_2$; $\Sigma_1, \Sigma'$; $\Sigma_2, \Sigma'$ respectively. The transformations which we have been considering can be looked upon as transforming the surfaces $\bar{\Sigma}_1, \bar{\Sigma}_2, \bar{\Sigma}_1', \bar{\Sigma}_2'$ into one another; we have consequently for these transformations a theorem of permutability.

We have seen that all the surfaces $\bar{\Sigma}_1$ corresponding to transformations of surfaces $\Sigma$ by means of the same values of the functions $m, \phi, w$ have a conjugate system in correspondence with the lines of curvature of the surfaces $\Sigma$; and the tangents to the curves of this system at corresponding points of two surfaces $\bar{\Sigma}_1$ are parallel. Hence the congruences formed by joining corresponding points of two of these surfaces $\bar{\Sigma}_1$ have their developable surfaces in correspondence with the lines of curvature of the surfaces $\Sigma$. We can, therefore, look upon the above transformations as transformations of such a congruence into another whose developables correspond to the developables of the former. Furthermore, there is a theorem of permutability for these congruence transformations.

The function $w_1'$ whose expression is given by (54) is a solution of the tangential LAPLACE equation (44) for $S_1$. Hence there is a surface $\Sigma_1$ determined by this value of $W_1$. In order to determine the surface $\Sigma$ of which it is the transform by means of $m_1, \phi_1, w_1$, we substitute its value (54) in equation (58) and find that

$$W = (m_2 - m_1) w_2.$$

In like manner it is found that the surface $\Sigma$ determined by

$$W = (m_1 - m_2) w_1$$

is transformed by means of the functions $m_2, \phi_2, w_2$ into the surface $\Sigma_2$ whose tangent plane is at the distance $w_2'$ from the origin.

It is readily found from (48) that the radius of the spheres whose envelope is formed of the surfaces $\Sigma$ and $\Sigma_1$ is given by $R_1 = m_2 \Phi_2 / w_1$, and for $\Sigma$ and $\Sigma_2$ $R_2 = m_1 \Phi_1 / w_2$.

## § 7. *Groups of eight surfaces* $\Sigma$.

Given a surface $\Sigma$ and three surfaces $\Sigma_1, \Sigma_2, \Sigma_3$ transforms of it by transformations $T_1(w_1, \phi_1, m_1)$, $T_2(w_2, \phi_2, m_2)$, $T_3(w_3, \phi_3, m_3)$. In consequence of the theorem of permutability we can find without quadrature surfaces of the same kind $\Sigma_{12}, \Sigma_{23}, \Sigma_{31}$, which are transforms of the respective pairs of surfaces $(\Sigma_1, \Sigma_2)$, $(\Sigma_2, \Sigma_3)$, $(\Sigma_3, \Sigma_1)$. Thus a surface $\Sigma_{ik}$ is the transform of $\Sigma$ by the successive application of transformations $T_i, T_k'$ or $T_k, T_i'$; consequently $\Sigma_{ik}$ is the same surface as $\Sigma_{ki}$. Consider now three surfaces $\Sigma_i, \Sigma_{ik}, \Sigma_{il}$, where $k \neq l \neq i$; the surfaces $\Sigma_{ik}$ and $\Sigma_{il}$ are supposed to be obtained from $\Sigma_i$ by transformations $T_k', T_l'$ respectively. By the theorem of permutability there is a fourth surface $\Sigma_{ikl}$ which is the transform of $\Sigma_{ik}$ by a transformation $T_l''$ and

of $\Sigma_{il}$ by a transformation $T''_k$. We are now going to show that all the surfaces obtained as $\Sigma_{ikl}$ and corresponding to a permutation of the subscripts $i, k, l$ are one and the same surface.

For the sake of brevity we put

$$
\begin{aligned}
L_{12} &= e^{-2\theta}\left(\frac{\partial\phi_1}{\partial u}\frac{\partial\phi_2}{\partial u}+\frac{\partial\phi_1}{\partial v}\frac{\partial\phi_2}{\partial v}\right)+w_1w_2,\\
L_{23} &= e^{-2\theta}\left(\frac{\partial\phi_2}{\partial u}\frac{\partial\phi_3}{\partial u}+\frac{\partial\phi_2}{\partial v}\frac{\partial\phi_3}{\partial v}\right)+w_2w_3,\\
L_{31} &= e^{-2\theta}\left(\frac{\partial\phi_3}{\partial u}\frac{\partial\phi_1}{\partial u}+\frac{\partial\phi_3}{\partial v}\frac{\partial\phi_1}{\partial v}\right)+w_3w_1.
\end{aligned}
\tag{66}
$$

Evidently $L_{ik}=L_{ki}$, for $k\neq i$.

Denote by $(m_1,\ w_{21},\ \phi_{21})$ and $(m_3,\ w_{23},\ \phi_{23})$ the functions determining the transformations which change $\Sigma_2$ into two surfaces $\Sigma_{21}$ and $\Sigma_{23}$ respectively. From (54) and (55) it follows that

$$
\begin{aligned}
w_{21} &= \frac{L_{12}-m_1w_2\phi_1-m_2w_1\phi_2}{\phi_2}, \qquad \phi_{21}=\frac{L_{12}-m_1w_1\phi_2-m_2w_2\phi_1}{w_2},\\
w_{23} &= \frac{L_{23}-m_3w_2\phi_3-m_2w_3\phi_2}{\phi_2}, \qquad \phi_{23}=\frac{L_{23}-m_3w_3\phi_2-m_2w_2\phi_3}{w_2}.
\end{aligned}
\tag{67}
$$

If we write the linear element of the spherical representation of $\Sigma_{21}$ in the form

$$ds'^2_{12}=e^{-2\theta_{12}}(du^2+dv^2),$$

it follows from (53) that

$$e^{-\theta_{12}}=e^{\theta_2}\frac{w_{21}}{\phi_{21}}=e^{-\theta}\frac{L_{12}-m_1w_2\phi_1-m_2w_1\phi_2}{L_{12}-m_1w_1\phi_2-m_2w_2\phi_1}. \tag{68}$$

We denote by $\Sigma_{213}$, or $\Sigma_{231}$, the surface which forms with $\Sigma_2$, $\Sigma_{21}$, $\Sigma_{23}$ a quatern and write the linear element of its spherical representation in the form

$$ds'^2_{213}=e^{-2\theta_{213}}(du^2+dv^2).$$

From (68) and (53) it follows that

$$e^{-\theta_{213}}=e^{-\theta_2}\frac{K}{J}=e^{\theta}\frac{w_2K}{\phi_2J}, \tag{69}$$

where we have put for the sake of brevity

$$
\begin{aligned}
J &= e^{-2\theta_2}\left(\frac{\partial\phi_{21}}{\partial u}\frac{\partial\phi_{23}}{\partial u}+\frac{\partial\phi_{21}}{\partial v}\frac{\partial\phi_{23}}{\partial v}\right)+w_{21}w_{23}-m_1w_{21}\phi_{23}-m_3w_{23}\phi_{21},\\
K &= e^{-2\theta_2}\left(\frac{\partial\phi_{21}}{\partial u}\frac{\partial\phi_{23}}{\partial u}+\frac{\partial\phi_{21}}{\partial v}\frac{\partial\phi_{23}}{\partial v}\right)+w_{21}w_{23}-m_1w_{23}\phi_{21}-m_3w_{21}\phi_{23}.
\end{aligned}
\tag{70}
$$

From (67) we get by differentiation

$$
\begin{aligned}
-e^{-\theta_2}\frac{\partial\phi_{21}}{\partial u} &= \frac{L_{12}-m_1w_1\phi_2-m_1w_2\phi_1}{\phi_2w_2}e^{-\theta}\frac{\partial\phi_2}{\partial u}+(m_1-m_2)e^{-\theta}\frac{\partial\phi_1}{\partial u},\\
e^{-\theta_2}\frac{\partial\phi_{21}}{\partial v} &= \frac{L_{12}-m_1w_1\phi_2-m_1w_2\phi_1}{\phi_2w_2}e^{-\theta}\frac{\partial\phi_2}{\partial v}+(m_1-m_2)e^{-\theta}\frac{\partial\phi_1}{\partial v},\\
-e^{-\theta_2}\frac{\partial\phi_{23}}{\partial u} &= \frac{L_{23}-m_3w_3\phi_2-m_3w_2\phi_3}{\phi_2w_2}e^{-\theta}\frac{\partial\phi_2}{\partial u}+(m_3-m_2)e^{-\theta}\frac{\partial\phi_3}{\partial u},\\
e^{-\theta_2}\frac{\partial\phi_{23}}{\partial v} &= \frac{L_{23}-m_3w_3\phi_2-m_3w_2\phi_3}{\phi_2w_2}e^{-\theta}\frac{\partial\phi_2}{\partial v}+(m_3-m_2)e^{-\theta}\frac{\partial\phi_3}{\partial v}.
\end{aligned}
\tag{71}
$$

Substituting these values and those from (67) in (70), we get

$$
\begin{aligned}
\phi_2J &= (m_1-m_2)(m_1-m_3)\phi_1L_{23}+(m_2-m_3)(m_2-m_1)\phi_2L_{31}\\
&\quad+(m_3-m_1)(m_3-m_2)\phi_3L_{12}-m_1(m_2-m_3)^2w_1\phi_2\phi_3\\
&\quad-m_2(m_3-m_1)^2w_2\phi_3\phi_1-m_3(m_1-m_2)^2w_3\phi_1\phi_2,\\
w_2K &= (m_1-m_2)(m_1-m_3)w_1L_{23}+(m_2-m_3)(m_2-m_1)w_2L_{31}\\
&\quad+(m_3-m_1)(m_3-m_2)w_3L_{12}-m_1(m_2-m_3)^2\phi_1w_2w_3\\
&\quad-m_2(m_3-m_1)^2\phi_2w_3w_1-m_3(m_1-m_2)^2\phi_3w_1w_2.
\end{aligned}
\tag{72}
$$

Since these expressions are symmetric in the functions, $(m_1, \phi_1, w_1)$, $(m_2, \phi_2, w_2)$, $(m_3, \phi_3, w_3)$, it follows from (69) that all the surfaces $\Sigma_{ikl}$, for any permutation of the subscripts $i, k, l$, have the same representation of their lines of curvature. It remains for us to show that the function $W_{ik}$, which determines the distance from the origin to the tangent plane to $\Sigma_{ikl}$, is likewise symmetric in the above functions, in order to prove that all of these surfaces $\Sigma_{ikl}$ coincide.

As in (63) we put

$$
\begin{aligned}
\Omega'_{21} &= L_{12}-m_1w_1\phi_2-m_2w_2\phi_1, \qquad \Omega_{21} = L_{12}-m_1\phi_1w_2-m_2\phi_2w_1,\\
\Omega'_{23} &= L_{23}-m_3w_3\phi_2-m_2w_2\phi_3, \qquad \Omega_{23} = L_{23}-m_3\phi_3w_2-m_2\phi_2w_3.
\end{aligned}
\tag{73}
$$

Hence by (65) the functions $W_{21}$ and $W_{23}$ determining the surfaces $\Sigma_{21}$ and $\Sigma_{23}$ are given by

$$
\begin{aligned}
W_{21}\Omega'_{21} &= (m_1-m_2)(\phi_1W_1w_2-\phi_2W_2w_1)+\Omega_{21}W,\\
W_{23}\Omega'_{23} &= (m_3-m_2)(\phi_3W_3w_2-\phi_2W_2w_3)+\Omega_{23}W.
\end{aligned}
\tag{74}
$$

By analogy we have that $W_{213}$ is such that

$$W_{213}J=(m_3-m_1)(\phi_{23}W_{23}w_{21}-\phi_{21}W_{21}w_{23})+W_2K,$$

which reduces to

$$
\begin{aligned}
W_{213}\phi_2J &= (m_1-m_2)(m_1-m_3)W_1[L_{23}-m_2\phi_2w_3-m_3\phi_3w_2]\\
&\quad+(m_2-m_3)(m_2-m_1)W_2[L_{31}-m_3\phi_3w_1-m_1\phi_1w_3]\\
&\quad+(m_3-m_1)(m_3-m_2)W_3[L_{12}-m_1\phi_1w_2-m_2\phi_2w_1].
\end{aligned}
$$

Since by (72) $\phi_2 J$ is symmetric in the functions $(m_1, \phi_1, w_1)$, $(m_2, \phi_2, w_2)$, $(m_3, \phi_3, w_3)$ and the right-hand number is evidently of this kind, it is true also of $W_{213}$. Hence $\Sigma_{213}$ is a transform of $\Sigma_{21}$ in two ways by means of transformations involving $m_3$; and in like manner it is a transform of $\Sigma_{23}$ and $\Sigma_{31}$, each in two ways, by means of transformations involving the constants $m_1$ and $m_2$ respectively.

Thus we have groups of eight surfaces, with isothermal representation of their lines of curvature, each of which is related to three others by transformations involving three different constants, the constants being the same three for each member of the group.

An immediate consequence of this result is the theorem.*

*Given four surfaces $\Sigma$, $\Sigma_1$, $\Sigma_2$, $\Sigma_{12}$ forming a quatern; each of these surfaces can be transformed by a transformation involving a constant $m_3$ different from $m_1$ and $m_2$ such that the new surfaces form a quatern also.*

## § 8. *Surfaces with plane lines of curvature in both systems.*

Surfaces whose lines of curvature in both systems are plane have isothermal representation of these lines. For the curves upon the sphere must be circles and consequently have constant geodesic curvature. Hence they form an isothermal system.† Moreover, if the curves in one family of an isothermal system have constant geodesic curvature, the same is true of the curves in the other family. Hence the surfaces which we are about to consider are the only surfaces with isothermal representation with plane lines of curvature.

An orthogonal system of circles on the sphere is obtained, in the most general manner, by intersecting the sphere with two pencils of planes whoses axes are polar reciprocal with respect to the sphere.‡ There are two cases; 1°, when the axes are tangent to the sphere; 2°, when the axes are not tangent.

1°. By taking for axes the two tangents to the sphere at the point $(0, 0, 1)$, which are parallel to the axes $Ox$, $Oy$, the coördinates of the sphere can be given in the form

$$(75) \qquad X = \frac{2u}{u^2+v^2+1}, \qquad Y = \frac{2v}{u^2+v^2+1}, \qquad Z = \frac{u^2+v^2-1}{u^2+v^2+1},$$

from which we get for the linear element

$$(76) \qquad ds'^2 = \frac{4(du^2+dv^2)}{(u^2+v^2+1)^2}.$$

2°. If the axes are parallel to the axes $Ox$, $Oy$, and meet the axis $Oz$ in the

* BIANCHI (loc. cit., p. 127) has established a similar theorem for the transformations $D_m$.

† BIANCHI, *Lezioni*, I, p. 210; German translation, p. 177.

‡ Ibid. I, p. 108; German translation, p. 81.

points $(0, 0, 1/a)$, $(0, 0, a)$, the coördinates of the sphere are expressible thus

$$(77)\quad X=\frac{\sqrt{1-a^2}\sin u}{\cosh v+a\cos u},\qquad Y=\frac{\sqrt{1-a^2}\sinh v}{\cosh v+a\cos u},\qquad Z=\frac{\cos u+a\cosh v}{\cosh v+a\cos u},$$

and the linear element is given by

$$(78)\qquad ds'^2=\frac{(1-a^2)(du^2+dv^2)}{(\cosh v+a\cos u)^2}.$$

From (76) and (78) it is seen that in either case $e^\theta$ is the sum of functions of $u$ and $v$ alone, so that equation (41) reduces to

$$\frac{\partial^2\psi}{\partial u\,\partial v}=0.$$

Consequently the most general solution of equation (9) is

$$(79)\qquad W=(U+V)e^{-\theta},$$

where $U$ and $V$ are arbitrary functions of $u$ and $v$ alone.

We denote by $U_1$ and $V_1$ the values of these functions which give a function $w$ determining a THYBAUT transformation; thus

$$(80)\qquad w=(U_1+V_1)e^{-\theta}.$$

We consider the first case (75). Now

$$(81)\qquad e^\theta=\frac{u^2+v^2+1}{2},$$

and equations $(A)$ reduce to

$$\frac{\partial\phi}{\partial u}=\frac{u^2+v^2+1}{2}U_1'-u(U_1+V_1),$$

$$\frac{\partial\phi}{\partial v}=-\frac{u^2+v^2+1}{2}V_1'+v(U_1+V_1).$$

If these values be substituted in equations $(B)$ and $(C)$, it is found that

$$m\phi=\frac{u^2+v^2+1}{2}m(U_1-V_1)-(uU_1'+vV_1')+(U_1+V_1),$$

and

$$(82)\qquad U_1=a\cosh\sqrt{2m}\,u,\qquad V_1=\pm a\cos\sqrt{2m}\,v,$$

where $a$ is an arbitrary constant.

When, in the second case,

$$(83)\qquad e^\theta=\frac{(\cosh v+a\cos u)}{\sqrt{1-a^2}},$$

equations $(A)$ reduce to

$$\frac{\partial \phi}{\partial u} = e^{\theta} U_1' + (U_1 + V_1) \frac{a \sin u}{\sqrt{1 - a^2}},$$

$$\frac{\partial \phi}{\partial v} = - e^{\theta} V_1' + (U_1 + V_1) \frac{\sinh v}{\sqrt{1 - a^2}}.$$

Proceeding as in the former case we find that we must consider separately the case where $m = \frac{1}{2}$.

When $m \neq \frac{1}{2}$, we get

$$2m\phi = \frac{1}{\sqrt{1 - a^2}} [(\cosh v + a \cos u)(U_1 - V_1) + 2(a \sin u U_1' - \sinh v V_1') + (U_1 + V_1)(\cosh v - a \cos u)],$$

and

$$U_1 = a \cosh (\sqrt{2m - 1}\, u), \qquad V_1 = \pm a \cos (\sqrt{2m - 1}\, v). \tag{84}$$

For the particular case $m = \frac{1}{2}$ we find

$$\phi = 4\alpha e^{\theta} + \frac{2}{\sqrt{1 - a^2}} (a \sin u U_1' - \sinh v V_1') + \frac{(U_1 + V_1)}{\sqrt{1 - a^2}} (\cosh v - \cos u)$$

and

$$U_1 = \alpha u^2 + a_1 u + a_2, \qquad V_1 = \alpha v^2 + b_1 v + b_2, \tag{85}$$

where the constants of integration must satisfy the relation

$$a_1^2 + b_1^2 = 4\alpha (a_2 + b_2) \tag{86}$$

in order that equation $(C)$ be satisfied.

When these values of $\phi$ and $w$ are substituted in (45) and $W$ is replaced by its expression (79), the function $W_1$ determining the transform $\Sigma_1$ of the given surface $\Sigma$ is given by quadratures. We can thus find all the transforms of surfaces with plane lines of curvature in both systems and from the theorem of permutability it follows that all the transformations of these new surfaces are given without quadrature.

We inquire now whether any of these transforms $\Sigma_1$ have plane lines of curvature in both systems. From (27) and (80) it follows that the linear element of the spherical representation of the surfaces $\Sigma_1$ is

$$ds_1'^2 = \frac{(U_1 + V_1)^2}{\phi^2} (du^2 + dv^2). \tag{87}$$

Hence the necessary and sufficient condition that the lines of curvature be plane in both systems is that

$$\frac{\partial^2}{\partial u \partial v} \left( \frac{\phi}{U_1 + V_1} \right) = 0.$$

One finds that for both cases (81) and (83) this equation reduces to $\phi U_1' V_1' = 0$. Hence either $U_1$ or $V_1$ is a constant.

Let $U_1$ be a constant. This is not possible for the cases (82) and (84). However, it is possible for (85) by taking $\alpha = a_1 = 0$; it follows from (86) that $b_1 = 0$ also. Hence $V_1$ is a constant. Without loss of generality we can take $U_1 + V_1 = 1$. Now

$$\phi = \frac{\cosh v - a \cos u}{\sqrt{1 - a^2}}. \tag{88}$$

Then the linear element (87) becomes

$$ds_1'^2 = \frac{(1 - a^2)(du^2 + dv^2)}{(\cosh v - a \cos u)^2}, \tag{89}$$

and by means of (25) it is found that

$$X' = -\frac{\sqrt{1 - a^2} \sin u}{\cosh v - a \cos u}, \quad Y' = \frac{\sqrt{1 - a^2} \sinh v}{\cosh v - a \cos u}, \quad Z' = -\frac{\cos u - a \cosh v}{\cosh v - a \cos u}. \tag{90}$$

Equations (45) reduce in the present case to

$$\frac{\partial}{\partial u}(\phi W_1) = -U', \qquad \frac{\partial}{\partial v}(\phi W_1) = V',$$

of which the integral is

$$\phi W_1 = -(U - V) + 2c_1,$$

where $c_1$ is a constant. The expression (48) for the radius $R$ of the sphere tangent to $\Sigma$ and $\Sigma_1$ becomes

$$R = \frac{1}{\sqrt{1 - a^2}}[a(\sin u \cdot U' - \cos u \cdot U) - \sinh v \cdot V' + \cosh v \cdot V + c_1(\cosh v + a \cos u].$$

The functions $A$ and $B$ defined by (49′) have now the forms

$$A = (U'' + U - c_1)e^{-\theta}, \qquad B = (V'' - V - c_1)e^{-\theta}.$$

From (49) it follows that the coördinates of the center of the sphere are

$$\bar{\xi} = \int (U'' + U - c_1) \cos u \, du, \qquad \bar{\eta} = \int (V'' - V - c_1) \cosh v \, dv,$$

$$\bar{\zeta} = \frac{1}{\sqrt{1 - a^2}} \int -(U'' + U - c_1) \sin u \, du + a(V'' - V - c_1) \sinh v \, dv.$$

Hence the locus of the centers of the spheres is a surface of translation whose generating curves lie in planes parallel to the coördinate planes $\bar{\xi} = 0$, $\bar{\eta} = 0$. Moreover, the most general surfaces of translation of this kind can be defined by equations of the above form. By repeating the steps in inverse order we can establish the theorem: *Given a surface of translation generated by the*

*motion of a plane curve when a point of the latter describes a second plane curve whose plane is perpendicular to the plane of the former; it is the locus of the centers of a family of spheres which envelope two surfaces with plane lines of curvature in both families.*

When $a = 0$ in (77), the curves $u = \text{const.}$ on the sphere are great circles with a common diameter and the curves $v = \text{const.}$ are their parallels. From (89) and (90) it is seen that the same curves serve for the representation of $\Sigma$ and $\Sigma_1$, the correspondence being given by

$$X' = -X, \qquad Y' = Y, \qquad Z' = -Z.$$

Now the surface $\bar{\Sigma}$ is a cylinder whose generators $u = \text{const.}$ are parallel to the $y$-axis. Moreover, the spheres whose centers lie on a generator have the same radius.

Returning to the general case, $a \neq 0$, we get from equations $(A)$ and (88)

$$w = \frac{\sqrt{1-a^2}}{\cosh v + a\cos u}.$$

The coördinates of the surface $\Sigma_0$ whose tangent plane is at the distance $w$ from the origin are

$$(91)\quad \xi = \frac{\sin u \cosh v}{\cosh v + a\cos u}, \qquad \eta = \frac{-a\cos u \sinh v}{\cosh v + a\cos u}, \qquad \zeta = \frac{\sqrt{1-a^2}\cos u \cosh v}{\cosh v + a\cos u}.$$

We find that the fundamental functions for this surface have the values

$$E, F, G = \frac{\cosh^2 v,\ 0,\ a^2\cos^2 u}{(\cosh v + a\cos u)^2},$$

$$D, D', D'' = \frac{-\sqrt{1-a^2}\cosh v,\ 0,\ \sqrt{1-a^2}\cdot a\cos u}{(\cosh v + a\cos u)^2}.$$

From these expressions it is seen that the centers of curvatures of the surface are defined by

$$x_1 = \xi - \frac{\cosh v}{\sqrt{1-a^2}}X, \qquad x_2 = \xi + \frac{a\cos u}{\sqrt{1-a^2}}X,$$

and similar expressions for the $y$ and $z$.

Replacing the various terms in the right-hand members by their values from (77) and (91), we get

$$x_1 = 0, \qquad y_1 = -\sinh v, \qquad z_1 = -\frac{a}{\sqrt{1-a^2}}\cosh v,$$

$$x_2 = \sin u, \qquad y_2 = 0, \qquad z_2 = \frac{\cos u}{\sqrt{1-a^2}}.$$

Hence when $a \neq 0$ the surface of centers consists of the two focal conics

$$x_1 = 0, \qquad (1-a^2)z_1^2 - a^2 y_1^2 = 1,$$
$$y_2 = 0, \qquad x_2^2 + (1-a^2)z_2^2 = 1.$$

And when $a = 0$ the surface of centers reduces to the $y$-axis and the circle

$$y_2 = 0, \qquad x_2^2 + z_2^2 = 1.$$

From this it follows that the surfaces $\Sigma_0$ are the cyclides of Dupin.

§ 9. *Surfaces* $\Sigma$ *with spherical lines of curvature in one system.*

ENNEPER* has shown that the necessary and sufficient condition that the lines of curvature $v = \text{const.}$ be spherical is that the following relation obtain

$$\sqrt{E} = R\cos\sigma \cdot \frac{\sqrt{E}}{\rho_1} + R\sin\sigma \frac{\rho_2}{\sqrt{G}} \frac{\partial}{\partial v} \frac{\sqrt{E}}{\rho_1},$$

where $\rho_1$ and $\rho_2$ are the principal radii of curvature, $R$ the radius of the sphere and $\sigma$ the angle under which the latter cuts the surface; both $R$ and $\sigma$ are functions of $v$ alone. For the surfaces $\Sigma$

$$\sqrt{E} = -\rho_1 e^{-\theta}, \quad \sqrt{G} = \rho_2 \cdot e^{-\theta},$$

so that the above equation becomes

$$\sqrt{E} = \alpha e^{-\theta} + \beta \frac{\partial \theta}{\partial v}, \tag{92}$$

where $\alpha$ and $\beta$ are functions of $v$ alone.

We consider first the case where $\Sigma$ is a minimal surface; now (92) becomes

$$e^{2\theta} = \alpha + \beta \frac{\partial}{\partial v} e^{\theta}.$$

If this be differented with respect to $u$, we get

$$\frac{\partial}{\partial u} e^{2\theta} = \beta \frac{\partial^2}{\partial u \partial v} e^{\theta}.$$

Eliminating $\beta$ from this equation and its derivative with respect to $u$, we get

$$\frac{\partial \theta}{\partial u} \frac{\partial^3 \theta}{\partial u^2 \partial v} - \frac{\partial^2 \theta}{\partial u^2} \frac{\partial^2 \theta}{\partial u \partial v} - \left(\frac{\partial \theta}{\partial u}\right)^2 \frac{\partial \theta}{\partial v} = 0. \tag{93}$$

This equation is satisfied by solutions of the equation

$$\frac{\partial^2}{\partial u \partial v} e^{\theta} = 0,$$

*Göttinger Nachrichten, 1868, p. 421; BIANCHI, *Lezioni*, II, p. 304.

that is, when the lines of curvature in both systems are plane. Excluding this case, every common solution of equations (3) and (93) determines a minimal surface whose lines of curvature $v = \text{const.}$ are spherical. DOBRINER * showed that there are minimal surfaces of this kind and found the expressions for their coördinates in the WEIERSTRASS form; they involve $\vartheta$-functions.

In a similar manner it can be shown that when $\theta$ is a solution of equation (3) and of

$$\frac{\partial\theta}{\partial v}\frac{\partial^3\theta}{\partial u\partial v^2} - \frac{\partial^2\theta}{\partial v^2}\frac{\partial^2\theta}{\partial u\partial v} - \frac{\partial\theta}{\partial u}\left(\frac{\partial\theta}{\partial v}\right)^3 = 0, \tag{94}$$

the lines of curvature $u = \text{const.}$ on the corresponding minimal surface are spherical.

In order to consider the case where $\Sigma$ is not a minimal surface we remark that the functions $E$ and $G$ are determined by the equations

$$\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v} = \frac{\partial\theta}{\partial v}, \qquad \frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u} = \frac{\partial\theta}{\partial u}.$$

If the expression (92) for $E$ be substituted in the first of these equations, we get

$$\sqrt{G} = \left(\alpha' e^{-\theta} + \beta\frac{\partial^2\theta}{\partial v^2}\right)\left(\frac{\partial\theta}{\partial v}\right)^{-1} + \beta' - \alpha e^{-\theta}, \tag{95}$$

where the accents denote differentiation with respect to $v$. Substituting this value in the second of the above equations, we get

$$\alpha' e^{-\theta}\left(\frac{\partial^2\theta}{\partial u\partial v} + \frac{\partial\theta}{\partial u}\frac{\partial\theta}{\partial v}\right) - \beta\left[\frac{\partial\theta}{\partial v}\frac{\partial^3\theta}{\partial u\partial v^2} - \frac{\partial^2\theta}{\partial u\partial v}\frac{\partial^2\theta}{\partial v^2} - \frac{\partial\theta}{\partial u}\left(\frac{\partial\theta}{\partial u}\right)^3\right] = 0. \tag{96}$$

Comparing this equation with (94) we see that if the lines of curvature $u = \text{const.}$ on the minimal surface $S$ are spherical, the lines of curvature $v = \text{const.}$ are spherical on the surfaces with the same representation as $S$ and for which $E$ and $G$ are given by

$$\sqrt{E} = \alpha e^{-\theta} + \beta\frac{\partial\theta}{\partial v}, \qquad \sqrt{G} = \beta' - \alpha e^{-\theta} + \beta\frac{\partial^2\theta}{\partial v^2}\left(\frac{\partial\theta}{\partial v}\right)^{-1} \tag{97}$$

where now $\alpha$ is any constant and $\beta$ any function whatever of $v$. When in particular $\alpha = 0$, it follows from (92) that the spheres upon which the curves $v = \text{const.}$ lie cut the surface orthogonally; and for any other value of $\alpha$ the projection of the radius of the sphere upon the tangent plane is constant.

If equation (96) be differentiated with respect to $u$ and $\alpha'$ and $\beta$ be eliminated

**Die Minimalflächen mit einem System sphärischer Krümmungslinien*, Acta Mathematica, vol. 10 (1887), pp. 145–152.

we get the equation of condition

$$(98)\qquad \begin{vmatrix} e^{-\theta}\left(\dfrac{\partial^2\theta}{\partial u\,\partial v}+\dfrac{\partial\theta}{\partial u}\dfrac{\partial\theta}{\partial v}\right), & \dfrac{\partial\theta}{\partial v}\dfrac{\partial^3\theta}{\partial u\,\partial v^2}-\dfrac{\partial^2\theta}{\partial u\,\partial v}\dfrac{\partial^2\theta}{\partial v^2}-\dfrac{\partial\theta}{\partial u}\left(\dfrac{\partial\theta}{\partial v}\right)^3 \\ \dfrac{\partial}{\partial u}\left[e^{-\theta}\left(\dfrac{\partial^2\theta}{\partial u\,\partial v}+\dfrac{\partial\theta}{\partial u}\dfrac{\partial\theta}{\partial v}\right)\right], & \dfrac{\partial}{\partial u}\left[\dfrac{\partial\theta}{\partial v}\dfrac{\partial^3\theta}{\partial v\,\partial u\,\partial v^2}-\dfrac{\partial^2\theta}{\partial u\,\partial v}\dfrac{\partial^2\theta}{\partial v^2}-\dfrac{\partial\theta}{\partial u}\left(\dfrac{\partial\theta}{\partial v}\right)^3\right] \end{vmatrix}=0.$$

Given a solution of this equation which is also a solution of equation (3), there is an infinity of surfaces $\Sigma$ with the given spherical representation of their lines of curvature and for which the curves $v=$ const. are spherical. The fundamental functions are given by (97) in which $\alpha$ is an arbitrary function of $v$ and $\beta$ is determined by (96).

In particular, if $\theta$ be a solution of (93) it satisfies (98); that is, when the curves $v=$ const. are spherical on a minimal surface there are an infinity of surfaces $\Sigma$ with the same spherical representation for which the curves $v=$ const. are spherical.

From (92) it follows that the coördinates $\xi$, $\eta$, $\zeta$ of a point on one of the surfaces $\Sigma$ with spherical lines of curvature $v=$ const. are given by quadratures of the form

$$\frac{\partial\xi}{\partial u}=\left(\alpha e^{-\theta}+\beta\frac{\partial\theta}{\partial v}\right)X_1=\frac{\partial}{\partial u}(\alpha X+\beta X_2);$$

hence

$$(99)\qquad \xi=\alpha X+\beta X_2+V_1,\qquad \eta=\alpha Y+\beta Y_2+V_2,\qquad \zeta=\alpha Z+\beta Z_2+V_3,$$

where $V_1$, $V_2$, $V_3$ are functions of $v$ alone. But from (95) we have

$$\frac{\partial\xi}{\partial v}=\left(\frac{\alpha' e^{-\theta}+\beta\dfrac{\partial^2\theta}{\partial v^2}}{\dfrac{\partial\theta}{\partial v}}+\beta'-\alpha e^{-\theta}\right)X_2.$$

Substituting the above value of $\xi$, we get

$$(100)\qquad V_1'=\beta\frac{\partial\theta}{\partial u}X_1+\frac{\alpha' e^{-\theta}+\beta\dfrac{\partial^2\theta}{\partial v^2}}{\dfrac{\partial\theta}{\partial u}}X_2-(\alpha'+\beta e^{-\theta})X.$$

It is readily verified that the expression on the right is a function of $v$ alone. Hence the functions $V_1$, $V_2$, $V_3$, and consequently the surfaces $\Sigma$, can be found by quadratures.

PRINCETON, *June*, 1906.

# THE EQUILONG TRANSFORMATIONS OF SPACE*

BY

JULIAN LOWELL COOLIDGE

In August, 1904, SCHEFFERS presented to the International Mathematical Congress in Heidelberg a paper entitled *Ueber Isogonalkurven, Aequitangentialkurven, und komplexe Zahlen.* This is published in the official account of the Congress † and again in much greater detail, but under the same title, in the Mathematische Annalen of the following year.‡ In these papers he has discussed a type of transformation of the plane which he has called "equilong" and which carries straight lines into straight lines, keeping invariant the distance between the points of contact of a line with any two of its envelopes. He has exhibited § the beautiful duality existing between these transformations and those of the conformal group; even as the latter depend upon an arbitrary function of the usual complex variable, so are the equilong transformations of the plane expressed by an arbitrary function of a complex variable of a different type. There seems no *a priori* reason why this close analogy of conformal and equilong transformations should not subsist in three or more dimensions. Such is not, however, the case. In December, 1904, STUDY stated, in a short article, *Ueber mehrere Probleme der Geometrie, die dem Problem der konformen Abbildung analog sind*: ‖ "Alle diese Aufgaben, deren beide letzte Analoga zum Problem der konformen Abbildung sind, lassen sich durch explizite Formeln lösen, . . . Man findet in allen Fällen unendliche Gruppen. Die Mitteilung der zugehörigen Formeln, die nicht in der Art mit hyperkomplexen Grössen zusammenzuhängen scheinen . . . wollen wir bis zu einer ausführlicheren Darstellung aufschieben." Since these words were written, three years ago, nothing further has been published concerning equilong transformations of space. The object of the present paper, which, incidentally, was originally written in ignorance of Professor STUDY's work, is to give an analytic discussion of the problem and to demonstrate the remarkable theorem which he indicates, namely, that,

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† Verhandlungen des dritten internationalen Mathematiker-Kongresses, Leipzig, 1905.

‡ Vol. 60 (1905).

§ Mathematische Annalen, loc. cit., p. 528.

‖ Sitzungsberichte der Niederrheinischen Gesellschaft für Natur- und Heilkunde, December 5, 1904, p. 59.

whereas the conformal group in three dimensions depends on ten arbitrary parameters merely, the equilong group involves two arbitrary functions.

Let us define as "equilong" any analytic transformation of the assemblage of all oriented planes of space, which carries a plane into a plane, and preserves invariant the distance between the points of contact of a plane with any two envelopes which it may touch. We mean by "oriented plane," of course, a plane whose aspect, or the direction of whose normals, is well determined, and two oriented planes shall be considered as infinitely near when not only their cartesian coördinates, but their aspects differ infinitesimally. We may therefore restate our problem as follows: Consider an oriented plane, and three non-coaxial oriented planes infinitely near thereto. Then the triangle which these three determine in the original plane shall go over into an equal triangle determined in the same way by the corresponding planes. We shall take as coördinates of our oriented plane $a$, $b$, $c$, the direction cosines of its directed normal, and $p$ the distance from the origin; all quantities involved being supposed, for simplicity, to be real. The corresponding plane shall be determined by $A$, $B$, $C$ and $P$. It is clear from our definition that parallel planes must go into parallel planes; hence $A$, $B$, $C$ will be functions of $a$, $b$, $c$ only. Moreover, from the equality of the triangles just mentioned, we see that two planes infinitely near a given plane will cut in two lines whose angle is invariant. Therefore

$$\frac{da\,da' + db\,db' + dc\,dc'}{\sqrt{da^2+db^2+dc^2}\sqrt{da'^2+db'^2+dc'^2}} = \frac{dA\,dA' + dB\,dB' + dC\,dC'}{\sqrt{dA^2+dB^2+dC^2}\sqrt{dA'^2+dB'^2+dC'^2}}.$$

This equation is most easily fulfilled by looking upon our coördinates $a$, $b$, $c$, which satisfy the equation

$$a^2 + b^2 + c^2 = 1,$$

as representing points of a unit sphere, and seeking the most general conformal transformation thereof. We may therefore write

$$a = \frac{z+\bar{z}}{z\bar{z}+1}, \qquad b = \frac{i(z-\bar{z})}{z\bar{z}+1}, \qquad c = \frac{z\bar{z}-1}{z\bar{z}+1},$$

$$A = \frac{Z+\bar{Z}}{Z\bar{Z}+1}, \qquad B = \frac{i(Z-\bar{Z})}{Z\bar{Z}+1}, \qquad C = \frac{Z\bar{Z}-1}{Z\bar{Z}+1},$$

(1) $$Z = f(z), \qquad \bar{Z} = \bar{f}(\bar{z}),$$

(2) $$Z = f(\bar{z}), \qquad \bar{Z} = \bar{f}(z).$$

The distinction between cases (1) and (2) is that in the first the orienting coördinates are subjected to a direct, and in the second, to an inverse, conformal transformation. Let us also write, for the sake of later reductions, the

equations

$$da\,d'a + db\,d'b + dc\,d'c = \frac{2(dz\,d'\bar{z} + d'z\,d\bar{z})}{(z\bar{z}+1)^2},$$

$$dA\,d'A + dB\,d'B + dC\,d'C = \frac{2(dz\,d'\bar{z} + d'z\,d\bar{z})}{(f(z)\bar{f}(\bar{z})+1)^2} f'(z)\bar{f}'(\bar{z}).$$

Our problem is now solved so far as the direction cosines are concerned; it remains merely to find $P = P(a, b, c, p)$. We have already insured that the triangle cut in any oriented plane by the neighboring planes shall be carried into a similar triangle; all that remains is to arrange matters so that the area of this triangle shall be invariant in absolute value.

Let us start with the plane

$$ax + by + cz = p,$$

or $(a, b, c, p)$, and find the area of the triangle cut therein by the planes

$$(a + da, b + db, c + dc, p + dp),\ (a + d'a, b + d'b, c + d'c, p + d'p),$$
$$(a + d''a, b + d''b, c + d''c, p + d''p).$$

The most symmetrical, if not the most immediately obvious, way to proceed, is to find the coördinates of the point where the last three planes meet, to find its distance from the first plane, and to divide into the volume of the tetraedron formed by the four planes. Proceeding thus we find that the square root of the area of our triangle differs by a constant factor from

$$\frac{\begin{vmatrix} a & b & c & p \\ da & db & dc & dp \\ d'a & d'b & d'c & d'p \\ d''a & d''b & d''c & d''p \end{vmatrix}}{\sqrt{\begin{vmatrix} a & b & c \\ da & db & dc \\ d'a & d'b & d'c \end{vmatrix}}\ \sqrt{\begin{vmatrix} a & b & c \\ d'a & d'b & d'c \\ d''a & d''b & d''c \end{vmatrix}}\ \sqrt{\begin{vmatrix} a & b & c \\ d''a & d''b & d''c \\ da & db & dc \end{vmatrix}}}.$$

This is our second invariant. It should be noticed that the cofactor of $p$ in the numerator is zero, in view of the equations

$$\Sigma a\,da = \Sigma a\,da' = \Sigma a\,d''a = 0;$$

we have, therefore, but one type of expression in either numerator or denominator,

$$\begin{vmatrix} a & b & c \\ da & db & dc \\ d'a & d'b & d'c \end{vmatrix} = \sqrt{\begin{vmatrix} 1 & 0 & 0 \\ 0 & \Sigma da^2 & \Sigma da\,d'a \\ 0 & \Sigma da\,d'a & \Sigma d'a^2 \end{vmatrix}} = \frac{2i(dz\,d'\bar{z} - d'z\,d\bar{z})}{(z\bar{z}+1)^2}.$$

Our invariant will therefore differ by a numerical factor from

$$\frac{\begin{vmatrix} dp & d'p & d''p \\ dz & d'z & d''z \\ d\bar{z} & d'\bar{z} & d''\bar{z} \end{vmatrix} (z\bar{z}+1)}{\sqrt{(dzd'\bar{z}-d'zd\bar{z})(d'zd''\bar{z}-d''zd\bar{z})(d''zdz-dzd''\bar{z})}}$$

$$= \frac{\begin{vmatrix} dP & d'P & d''P \\ dz & d'z & d''z \\ d\bar{z} & d'\bar{z} & d''\bar{z} \end{vmatrix} [f(z)\bar{f}(\bar{z})+1]}{\sqrt{f'(z)\bar{f}'(\bar{z})}\sqrt{(dzd'\bar{z}-d'zd\bar{z})(d'zd''\bar{z}-d''zd'\bar{z})(d''zdz-dzd''\bar{z}}}$$

$$= \frac{\dfrac{\partial P}{\partial p}\begin{vmatrix} dp & d'p & d''p \\ dz & d'z & d''z \\ d\bar{z} & d'\bar{z} & d''\bar{z} \end{vmatrix} [f(z)\bar{f}(\bar{z})+1]}{\sqrt{f'(z)\bar{f}'(\bar{z})}\sqrt{\begin{vmatrix} dz & d\bar{z} \\ d'z & d'\bar{z} \end{vmatrix}}\sqrt{\begin{vmatrix} d'z & d'\bar{z} \\ d''z & d''\bar{z} \end{vmatrix}}\sqrt{\begin{vmatrix} d''z & d''\bar{z} \\ dz & d\bar{z} \end{vmatrix}}},$$

the last reduction being effected by means of the formula

$$dP = \frac{\partial P}{\partial p}dp + \frac{\partial P}{\partial z}dz + \frac{\partial P}{\partial \bar{z}}d\bar{z}.$$

We thus reach the simple differential equation

$$\frac{\partial P}{\partial p} = \frac{(z\bar{z}+1)}{[f(z)\bar{f}(\bar{z})+1]}\sqrt{f'(z)\bar{f}(z)},$$

and this yields the final forms for our transformation:

$$P = \frac{(z\bar{z}+1)p}{[f(z)\bar{f}(\bar{z})+1]}\sqrt{f'(z)\bar{f}'(\bar{z})} + Q(z, z),$$

(I)

(1) $Z = f(z), \qquad \bar{Z} = \bar{f}(\bar{z}),$

(2) $Z = f(\bar{z}), \qquad \bar{Z} = \bar{f}(z).$

These equations may also be written in terms of real coördinates as follows:

$$z = x + iy, \qquad Z = X + iY,$$

$$X = \phi(x, y), \qquad Y = \psi(x, y).$$

$$P = \frac{(x^2+y^2+1)p}{(\phi^2+\psi^2+1)}\sqrt{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial x}\right)^2} + \chi(x, y),$$

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \qquad \frac{\partial \psi}{\partial x} = -\frac{\partial \phi}{\partial y}, \tag{1}$$

$$\frac{\partial \phi}{\partial x} = -\frac{\partial \psi}{\partial y}, \qquad \frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y}. \tag{2}$$

It is easy to verify analytically the geometrically evident fact that our equilong transformations form a group.

It is clear that our method of reasoning may be carried at once into $n$ dimensions. If we call the parameters corresponding to $a$, $b$, $c$ and to $p$ respectively the direction and the distance parameters, we have

*The most general equilong transformation of a euclidean space of $n$ dimensions depends on the most general conformal transformation of a space of $n-1$ dimensions and an arbitrary function of the direction parameters. The distance parameter enters linearly.*

We see from this that in a space of more than three dimensions there is but one arbitrary function involved. It is well to insist on the word "euclidean" for in Riemannian or Lobatschewskian space, conformal and equilong transformations are identical. For this reason, the differences here noted for the euclidean case seem the more striking.

CAMBRIDGE, *October*, 1907.

# CONCERNING LINEAR SUBSTITUTIONS OF FINITE PERIOD WITH RATIONAL COEFFICIENTS*

BY

ARTHUR RANUM

## INTRODUCTION.

If the coefficients of a linear homogeneous substitution (matrix, bilinear form) are unrestricted, i. e., belong to the continuous domain of all complex numbers, there is no upper limit to its finite period;† moreover, for all possible integral values of $n$ and $m$ there exist linear substitutions in $n$ variables of period $m$. But if the coefficients are restricted to the domain of rational numbers, the finite period is limited to certain definitely prescribed values that depend on the number of variables. For instance, it is clear that in one variable the only linear homogeneous substitutions of finite period with rational coefficients are $x' = x$, of period one, and $x' = -x$, of period two. Again, in two variables the only finite periods will be shown to be 1, 2, 3, 4, and 6. This may be compared with the well-known result given by KLEIN and FRICKE‡ that the finite periods of linear *fractional* substitutions of *determinant* 1 in one variable are 1, 2, and 3.

In the continuous domain of complex numbers, every linear substitution of finite period is completely reducible; i. e., its canonical form is axial; but in the domain of rational numbers, some linear substitutions are only partly reducible and some are entirely irreducible; hence their canonical forms are no longer so simple. Moreover, there is an intimate connection between the reducibility of a linear substitution and its period.

In §§ 1–4 of this paper the minimum degree (number of variables) of a linear substitution of given period $m$ with rational coefficients is determined;

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† In this paper I shall confine my attention to matrices whose determinants do not vanish and therefore define the period of a matrix and the identity matrix in the usual way; in another paper soon to be published elsewhere I shall consider groups of matrices whose determinants vanish, and arrive at a more general definition of the concepts *period* and *identity matrix*.

‡ *Modulfunktionen*, vol. 1, p. 182.

in §§ 5–9 the effect of its reducibility is found; in §§ 10–20 the inverse problem is solved of obtaining the values of the period of a linear substitution when its degree and reducibility are given; §§ 21–27 contain a method of constructing the canonical forms of all linear substitutions of finite period. The principal results are: (1) a complete determination of all the finite periods of $n$-ary linear substitutions having rational coefficients; (2) a precise classification of all $n$-ary linear substitutions of period $m$ having rational coefficients, with respect to their reducibility and canonical forms. Finally, § 28 contains a brief summary of these results for linear substitutions of low degree.

## THE MINIMUM DEGREE OF A MATRIX OF PERIOD $m$.

1. Let

$$L = (l_{ij}) = \begin{pmatrix} l_{11} \cdots l_{1n} \\ \cdots \\ l_{n1} \cdots l_{nn} \end{pmatrix}, \qquad \text{where} \qquad |l_{ij}| \neq 0,$$

be an $n$-ary matrix of period $m$ whose elements are rational numbers, and let its characteristic equation be written in the form

$$\Phi(\lambda) \equiv |l_{ij} - \lambda\delta_{ij}| = 0, \qquad \text{where} \qquad \delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

If we denote the identity matrix $(\delta_{ij})$ by $I$, then $L$ satisfies the two symbolic equations $L^m = I$ and $\Phi(L) = 0$, whose degrees are $m$ and $n$, respectively. FROBENIUS* has proved that a matrix is of finite period if, and only if, the elementary divisors of its characteristic determinant are all linear, and the roots of its characteristic equation are roots of unity. It follows that if its period is $m$, the L. C. M. of the orders of these roots of unity must be $m$.† Here the term *order* of a root of unity $\epsilon$ is defined as the lowest integer $\kappa$ for which $\epsilon^\kappa = 1$.

2. We shall need the following:

LEMMA. *Any rational, integral equation* $\Phi(\lambda) = 0$, *whose coefficients belong to a field* ‡ $\Omega$, *is the characteristic equation of some matrix whose elements belong to* $\Omega$.§

* Crelle's Journal, vol. 84 (1877), p. 16.

† It is incorrectly stated by MUTH (*Elementartheiler*, art. 91, p. 178) that at least one of these roots must be a primitive $m$-th root of unity.

‡ By the term *field* we shall understand, as usual, any set of elements that is closed under rational operations (excluding division by zero), while we shall confine the term *domain* to that particular kind of field whose elements, infinite in number, are ordinary real or complex numbers. The domain $(\varepsilon_1, \varepsilon_2, \cdots)$ will be understood to mean the set of rational functions of the numbers $\varepsilon_1, \varepsilon_2, \cdots$.

§ This lemma was given in substance by DICKSON in the American Journal of Mathematics, vol. 23 (1901), p. 37.

Proof. If $\Phi(\lambda) \equiv \lambda^n + d_1\lambda^{n-1} + \cdots + d_n$, let $L = (l_{ij})$ be chosen as defined by the equations

$$l_{i,i+1} = 1 \ (i = 1, \cdots, n-1), \qquad l_{n,j} = -d_{n-j+1} \ (j = 1, \cdots, n),$$

$$l_{ij} = 0 \ (j \neq i+1; \ i \neq n: \ i, j = 1, \cdots, n).$$

Then $L$ is evidently a matrix whose elements belong to $\Omega$, and whose characteristic equation is $\Phi(\lambda) = 0$, as required by the lemma.

Example. If $n = 3$,

$$\Phi(\lambda) \equiv \lambda^3 + d_1\lambda^2 + d_2\lambda + d_3, \qquad L = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -d_3 & -d_2 & -d_1 \end{pmatrix}.$$

3. Our object is to find, for any given value of the degree $n$ of a matrix $L$ with rational elements, all the values of its finite period $m$, which are not also periods of matrices with rational elements of degree lower than $n$; we shall attain this object by first obtaining, for any given value of its period $m$, its minimum degree $n$. Since the degree of a matrix is equal to the degree of its characteristic equation, and since by the above lemma every rational, integral equation with rational coefficients is the characteristic equation of some matrix with rational elements, therefore we wish to find an equation $\Phi(\lambda) = 0$, whose coefficients are rational, whose roots are roots of unity, such that the L. C. M. of their orders is $m$, whose elementary divisors are linear (when it is regarded as the characteristic equation of some definite matrix), and whose degree $n$ is a minimum.

As so defined, $\Phi(\lambda) = 0$ will not have multiple roots. If it had, then the equation

$$\frac{\Phi(\lambda)}{[\Phi(\lambda), \Phi'(\lambda)]} = 0,$$

where $[\Phi(\lambda), \Phi'(\lambda)]$ is the G. C. D. of $\Phi(\lambda)$ and its derivative, would not have multiple roots, and since its coefficients are rational, it would also be the characteristic equation of some matrix with rational elements; moreover, it would be satisfied by all the roots of $\Phi(\lambda) = 0$, its elementary divisors would be linear, and its degree would be less than that of $\Phi(\lambda) = 0$.

Suppose the function $\lambda^m - 1$ to be factored as far as possible in the domain (1). Corresponding to every divisor $m_1$ of $m$, there will be just one irreducible factor of $\lambda^m - 1$ that is annihilated by all the primitive $m_1$-th roots of unity and no others; and in particular there will be one irreducible factor $\Phi_1(\lambda)$, that is annihilated by all the primitive $m$-th roots of unity and no others. It is clear,

therefore, that our problem will be solved by putting $\Phi(\lambda)$ equal to the product of irreducible factors of $\lambda^m - 1$ selected in such a way that the sum of their degrees is a minimum and the G. C. D. of the orders of the corresponding roots of unity is equal to $m$. Its elementary divisors will evidently be linear.

4. If now we impose the further restriction that the characteristic determinant $\Phi(\lambda)$ shall be irreducible, we plainly have to put $\Phi(\lambda) = \Phi_1(\lambda)$; and since the degree of $\Phi_1(\lambda)$ is $\phi(m)$ (EULER'S $\phi$-function), the latter is not merely the minimum degree, but the only possible degree of the corresponding matrix. Thus we have derived

THEOREM 1. *Every matrix of period $m$ whose elements are rational and whose characteristic determinant is irreducible in the domain* (1) *is of degree $\phi(m)$. The characteristic determinant of every such matrix is that irreducible factor of $\lambda^m - 1$ which is annihilated by the primitive $m$-th roots of unity.*

Example. Every such matrix of period 12 is of degree 4, and its characteristic determinant is $\lambda^4 - \lambda^2 + 1$; a representative matrix is

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix}.$$

5. On the other hand, if we place no limitation on the reducibility of the characteristic equation, we can, of course, find matrices of period $m$ whose degree is higher than $\phi(m)$, but we can also in general find others of degree much lower than $\phi(m)$. Let $m = 2^a p_1^{a_1} \cdots p_r^{a_r}$, where $p_1, \cdots, p_r$ are the distinct odd prime factors of $m$ (if any exist), where $a_1, \cdots, a_r$ are positive integers, and where $a$ is a positive integer or zero. We wish to find a set of integers $m_1, \cdots, m_s$ whose L. C. M. is $m$, and the sum of whose $\phi$-functions is a minimum. It will evidently be sufficient to take these integers prime to each other. For if any two of them are not prime to each other, their G. C. D. is at most 2, and $\phi(2\kappa) = \phi(\kappa)$ where $\kappa$ is odd. Moreover, if $m_1$ and $m_2$ are prime to each other, $\phi(m_1 m_2) = \phi(m_1) \cdot \phi(m_2)$; and since the sum of two positive integers is less than, or equal to, their product, unless one of them is unity, it follows that $\phi(m_1) + \phi(m_2) \leqq \phi(m_1 m_2)$, unless $\phi(m_1)$, say, is 1, i. e., unless $m_1 = 2$.

Therefore, if $a > 1$, the required integers can be taken to be $2^a, p_1^{a_1}, \cdots, p_r^{a_r}$, and the required minimum degree of a matrix of period $m$ is $\phi(2^a) + \phi(p_1^{a_1}) + \cdots + \phi(p_r^{a_r})$; while if $a = 0$ or 1, the integers can be taken to be $2^a p_1^{a_1}, p_2^{a_2}, \cdots, p_r^{a_r}$, and the minimum degree is $\phi(p_1^{a_1}) + \cdots + \phi(p_r^{a_r})$.

If we define $\psi(m)$ by the equations

$$\psi(m) = \begin{cases} \phi(2^a) + \sum_{i=1}^{r} \phi(p_i^{a_i}), & \text{if } a > 1, \\ \sum_{i=1}^{r} \phi(p_i^{a_i}), & \text{if } a = 1 \text{ or } 0, \end{cases}$$

we have

THEOREM 2. *If the characteristic determinant of a matrix of period $m$, whose elements are rational, is reducible in the domain* (1), *its irreducible factors are those irreducible factors of $\lambda^m - 1$, which are annihilated respectively by the primitive $m_1$-th $\cdots$ $m_s$-th roots of unity, where $m_1, \cdots, m_s$ are integers whose L. C. M. is equal to $m$. The minimum degree of a matrix of period $m$ with rational elements is equal to $\psi(m)$.*

Example. If $m = 15$, the minimum degree is 6, and every matrix of period 15 and degree 6 has for its characteristic determinant

$$(\lambda^2 + \lambda + 1)(\lambda^4 + \lambda^3 + \lambda^2 + \lambda + 1),$$

a representative matrix is

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{pmatrix}.$$

## THE REDUCIBILITY OF MATRICES IN A GIVEN DOMAIN.

6. A classification of matrices with respect to their characteristic equations will now be considered, and the results will be obtained for matrices whose elements belong to any domain $\Omega$, not necessarily the domain (1), and not merely for matrices of finite period, but for all matrices whose characteristic determinants have linear elementary divisors.

If the totality of all the $n$-ary matrices in $\Omega$, whose determinants are not zero, is called the *chief linear group of* $\Omega$, then two such matrices, $L$ and $L'$, will be said to be *conjugate in* $\Omega$, if they are conjugate in the chief group of $\Omega$, i. e., if there exists a matrix $M$ in the chief group, such that $M^{-1}LM = L'$. A matrix $L$ will be said to be *reducible in* $\Omega$, if it is conjugate, in $\Omega$, to a matrix of the form

$$L' = \begin{pmatrix} L_1 & O_1 \\ O_2 & L_2 \end{pmatrix},$$

where $L_1$, $L_2$ are square matrices and $O_1$, $O_2$ are rectangular matrices whose elements are all zero. If the component matrices $L_1$ and $L_2$ are themselves irreducible, then $L'$ will be called a *reduced* matrix. If not, then $L'$ can be further transformed into a matrix having three or more components. When finally all the components are irreducible, the complete matrix is said to be *reduced* in $\Omega$. If the reduced matrix is axial, i. e., if all its elements outside of the principal diagonal are zero, the original matrix $L$ is said to be *completely reducible*. In the complete set of conjugates to a given matrix $L$ a simple representative matrix may be chosen as the *canonical form* of $L$. The canonical form of a reducible matrix will naturally be chosen to be a reduced matrix.

7. It is well known * that in the continuous domain of complex numbers all matrices, except those whose characteristic determinant has only a single elementary divisor, are reducible and that every matrix whose characteristic determinant has only linear elementary divisors is completely reducible. In a smaller domain, however, matrices are in general less reducible and the number of irreducible matrices is larger.

Example. The ternary matrix

$$L=\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{pmatrix},$$

of period 3, is completely reducible in the domain of complex numbers, and its reduced form is

$$\begin{pmatrix}1 & 0 & 0\\ 0 & \epsilon & 0\\ 0 & 0 & \epsilon^2\end{pmatrix},$$

where $\epsilon$ is a primitive cube root of unity; whereas in the domain (1) it is only partly reducible, and its canonical reduced form can be taken to be

$$L'=\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & -1 & -1\end{pmatrix}.$$

It is easy to verify that

$$\begin{pmatrix}1 & 0 & 1\\ 1 & -1 & -1\\ 1 & 1 & 0\end{pmatrix}$$

is a matrix that will transform $L$ into $L'$.

8. The following statement can be given to a well-known result in the theory of elementary divisors: †

* MUTH, *Elementartheiler*, arts. 77–79.

† FROBENIUS, Crelle's Journal, vol. 86 (1878), p. 146.

*Two matrices belonging to a domain $\Omega$ are conjugate in $\Omega$, if, and only if, the elementary divisors of their characteristic determinants coincide.*

In the particular case in which the elementary divisors are linear, they will coincide if the characteristic determinants themselves coincide. Therefore, *if the characteristic determinants of two matrices belonging to $\Omega$ have linear elementary divisors, then the coincidence of these characteristic determinants is the necessary and sufficient condition that the matrices are conjugate in $\Omega$.*

Among the matrices to which this theorem applies are those of finite period, since their elementary divisors are linear. Hence we see that *two matrices of finite period belonging to $\Omega$ are conjugate in $\Omega$, if, and only if, they have the same characteristic determinant.*

Example. The above-mentioned ternary matrices of period 3 both have the same characteristic determinant, $\lambda^3 - 1$.

9. We are now prepared to prove

THEOREM 3. *A matrix $L$, whose characteristic determinant $\Phi(\lambda)$ has its elementary divisors all linear, is reducible in a given domain $\Omega$, if, and only if, its characteristic determinant is reducible in that domain. If the irreducible factors of $\Phi(\lambda)$ are $\Phi_1(\lambda), \cdots, \Phi_s(\lambda)$, of degrees $n_1, \cdots, n_s$, respectively, then the reduced form of $L$ is made up of irreducible components $L_1, \cdots, L_s$, of degrees $n_1, \cdots, n_s$, respectively, whose characteristic determinants are $\Phi_1(\lambda), \cdots, \Phi_s(\lambda)$, respectively.*

Proof. (*a*) If $L$ is reducible in $\Omega$, it obviously has the same characteristic determinant as its reduced form. Moreover, the characteristic determinant of a reduced matrix is the product of the characteristic determinants of its components, and is therefore reducible in $\Omega$.

(*b*) If $\Phi(\lambda)$, the characteristic determinant of $L$, is reducible in $\Omega$, and its irreducible factors are $\Phi_1(\lambda), \cdots, \Phi_s(\lambda)$, then, by the lemma of § 2, component matrices $L_1, \cdots, L_s$ exist in $\Omega$ having the functions $\Phi_1(\lambda), \cdots, \Phi_s(\lambda)$ as their characteristic determinants. Moreover, the latter, being irreducible, have no multiple roots, and their elementary divisors are linear. Therefore the reduced matrix $L'$, made up of the components $L_1, \cdots, L_s$, has the same characteristic determinant, with the same linear elementary divisors, as $L$; i. e., $L$ and $L'$ are conjugate in $\Omega$ and $L$ is reducible in $\Omega$.

Example.

$$L = \begin{Vmatrix} 0 & 2 & 0 & 2 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{Vmatrix}$$

is a matrix whose characteristic determinant, $(\lambda^2-2)^2$, has the elementary divisors $(\lambda-\sqrt{2})$, $(\lambda-\sqrt{2})$, $(\lambda+\sqrt{2})$, and $(\lambda+\sqrt{2})$. Since this characteristic determinant is reducible in any domain, $L$ is reducible in any domain. In the continuous domain, $L$ is completely reducible and its canonical form is

$$\begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & -\sqrt{2} & 0 \\ 0 & 0 & 0 & -\sqrt{2} \end{bmatrix}.$$

In the domain (1), however, the irreducible factors of its characteristic determinant are $\lambda^2-2$ and $\lambda^2-2$ and its canonical form can be taken to be

$$\begin{bmatrix} 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

COROLLARY. *A matrix of finite period is reducible in a given domain, if, and only if, its characteristic determinant is reducible in that domain.*

Example. The reducibility of the matrix mentioned in § 7 is due to the reducibility of its characteristic determinant $(\lambda-1)(\lambda^2+\lambda+1)$.

## THE FINITE PERIODS OF $n$-ARY MATRICES.

10. In all that follows, the domain (1) will be understood.* By means of Theorem 3 and its corollary, the results of Theorems 1 and 2 can now be expressed in the following form:

THEOREM 4. *Every irreducible matrix of period $m$ with rational elements is of degree $\phi(m)$. Every reducible matrix of period $m$ with rational elements is of degree $\phi(m_1)+\cdots+\phi(m_s)$, where $m_1, \cdots, m_s$ are integers whose L. C. M. is equal to $m$; its reduced form is made up of irreducible components whose degrees are $\phi(m_1), \cdots, \phi(m_s)$, respectively. The minimum degree of a matrix of period $m$ with rational elements is $\psi(m)$.*

EXAMPLE. If $m=20$, the irreducible matrices are of degree 8; the reducible matrices are of various degrees ranging upward from the minimum value, which is 6; for instance, by taking $m_1=1$, $m_2=4$, $m_3=4$, $m_4=10$, we get $1+2+2+4=9$ for the degree.

11. Theorem 4 is analogous to the following well-known properties of permutations (substitutions):

* In a future paper I hope to treat in a similar manner the domain $(\varepsilon)$, where $\varepsilon$ is a root of unity.

Every transitive (cyclical) permutation of period $m$ is of degree $m$. Every intransitive permutation of period $m$ is of degree $m_1 + \cdots + m_s$, where $m_1, \cdots, m_s$ are integers whose L. C. M. is equal to $m$. The minimum degree of a permutation of period $2^a p_1^{a_1} \cdots p_r^{a_r}$ is $2^a + p_1^{a_1} + \cdots + p_r^{a_r}$, if $a > 0$, and $p_1^{a_1} + \cdots + p_r^{a_r}$, if $a = 0$.

EXAMPLE. A permutation of period 20 and of minimum degree 9 has one cycle of 4 letters and one of 5.

12. COROLLARY 1. *The minimum degree of a matrix whose period $m$ is a power of a prime, or twice a power of an odd prime, is $\phi(m)$. In particular, the minimum degree of a matrix of prime period $p$ is $p - 1$.*

From Corollary 1, in view of the lemma of § 2, we derive

COROLLARY 2. *If $p$ is a prime and $a$ is an integer determined by the inequalities $p^{a-1}(p-1) \leqq n < p^a(p-1)$, then $n$-ary matrices exist whose period is any power of $p$ as high as the $a$-th, but no higher. In particular, if $p - 1 \leqq n < p(p-1)$, $n$-ary matrices exist of period $p$, but none of period a power of $p$ higher than the first. The highest prime period of an $n$-ary matrix cannot be greater than $n + 1$.*

Since, for most values of $m$, $\psi(m) < \phi(m)$, it follows that in general the minimum degree of matrices of period $m$ is less than the degree of the irreducible matrices of that period; i. e., there exist reducible matrices of lower degree than the irreducible matrices of the same period.

But there are exceptional values of $m$ for which $\psi(m) = \phi(m)$. The solutions of this equation are evidently

$$(1)\quad m = 2^a, \qquad (2)\quad m = p^a, \qquad (3)\quad m = 2p^a, \qquad (4)\quad m = 12.$$

If $m$ has any of these values, the minimum degree of matrices of period $m$ is equal to the degree of the irreducible matrices of that period; and there are no reducible matrices of lower degree than the irreducible matrices of the same period.

Moreover, if $m = 2^a$, $p^a$, or $2p^a$, all the matrices of minimum degree $\phi(m)$ are irreducible. Hence, *every matrix whose period $m$ is a power of a prime or twice a power of an odd prime, and one whose degree is $\phi(m)$, is irreducible; the degree of every reducible matrix of the same period is greater than $\phi(m)$. In particular, every matrix of prime period $p$ and of degree $p - 1$ is irreducible; the degree of every reducible matrix of period $p$ is greater than $p - 1$.*

On the other hand, if $m = 12$, $\phi(m) = 2 \cdot 2$ and $\psi(m) = 2 + 2$. Therefore some of the matrices of period 12 and of degree 4 are irreducible and some are reducible; every reducible matrix of period 12 is of degree 4 or more.

13. With the single exception, $\phi(2) = \psi(2) = 1$, an odd number cannot be a

value of $\phi(m)$ or of $\psi(m)$. Moreover, not every even number can be a value of $\phi(m)$; among the even numbers which are not values of $\phi(m)$ the smallest are 14, 26, 34, 38, and 50. However, it is easy to see that every even number is included among the values of $\psi(m)$.

From these data drawn from the theory of numbers the following conclusions can be immediately drawn. *The degree of an irreducible matrix of finite period cannot be an odd number greater than* 1 *or one of the even numbers* 14, 26, 34, 38, 50, *etc.;* in other words, every matrix of finite period, whose degree is either an odd number greater than 1 or one of the even numbers 14, 26, 34, 38, 50, etc., is reducible. In particular, every ternary matrix of finite period is reducible in the domain (1)*.

*The minimum degree of a matrix of finite period can be any even number whatsoever, but cannot be an odd number greater than* 1; in other words, every even number is the minimum degree of the matrices of some finite period, whereas, if a matrix of finite period is of odd degree ($>1$), another matrix of the same period and of lower degree can be found.

14. Let $\phi^{-1}(x)$ and $\psi^{-1}(x)$ be defined in the usual way as the inverse $\phi$- and $\psi$-functions of $x$. As so defined they are multiple-valued for some integral values of $x$ and do not exist at all for other values of $x$. Neither $\phi^{-1}(x)$ nor $\psi^{-1}(x)$ exists for any odd value of $x$ greater than 1; $\phi^{-1}(x)$ exists for all even values of $x$ except 14, 26, etc., and $\psi^{-1}(x)$ exists for all even values of $x$ without exception.

The values of $\phi^{-1}(n)$ are evidently the finite periods of irreducible matrices of degree $n$; while the values of $\psi^{-1}(n)$ are those finite periods of reducible or irreducible matrices of degree $n$ which are not also periods of matrices of lower degree. To find all the finite periods of $n$-ary matrices, it is necessary, therefore, to include the totality of the values of $\psi^{-1}(x)$ for all integral values of $x$ from 1 to $n$ inclusive, i. e., if $n=2k$ or $2k+1$, to include the values of $\psi^{-1}(1)$, $\psi^{-1}(2)$, $\psi^{-1}(4), \cdots, \psi^{-1}(2k)$. In view of the lemma of § 2, we may state

THEOREM 5. *If $n=2k$ or $2k+1$, the finite periods of n-ary matrices with rational elements are the values of $\psi^{-1}(1), \psi^{-1}(2), \psi^{-1}(4), \cdots, \psi^{-1}(2k)$. There exist n-ary matrices having all these periods, but none having any other finite periods.*

COROLLARY. *The finite periods of matrices of degree $2k+1$ are exactly the same as those of degree $2k$.*

* On the other hand POINCARÉ has shown (*Les Fonctions fuchsiennes et l'arithmétique*, Liouville's Journal, ser. 4, vol. 3 (1887), pp. 439–449) that within the group of ternary matrices whose determinants are equal to unity and whose elements are not only rational but integral, the matrices of finite period are not all reducible, although their characteristic determinants evidently break up into factors with rational, and therefore integral, coefficients.

15. *Geometrical interpretation.* There are various ways in which $n$-ary matrices, or linear substitutions in $n$ variables, can be interpreted geometrically as collineations. For the present, let us regard the $n$ variables as cartesian point coördinates in flat space $R_n$ of $n$ dimensions. Then an $n$-ary matrix becomes an affine transformation of $R_n$ that leaves the origin fixed, an affine transformation being a collineation that leaves the $R_{n-1}$ at infinity fixed.

Now the corollary to Theorem 5 shows that *the finite periods of affine transformations with rational coefficients that leave the origin fixed are exactly the same in space of* $2k+1$ *dimensions as in space of* $2k$ *dimensions.* This result is analogous to certain other properties which distinguish space of an even number of dimensions from space of an odd number of dimensions. For example, in $R_{2k+1}$ as well as in $R_{2k}$ the most general movement about a fixed point is the resultant of $k$ independent rotations in $k$ mutually perpendicular planes.*

16. In order to compute with facility the values of $\phi^{-1}(n)$ and $\psi^{-1}(n)$ corresponding to a given value of $n$, it will be necessary first to construct a short auxiliary table of those particular values of $\phi^{-1}(n)$ which are powers of primes. These can be written down by inspection, as follows:

| $n$ | 1 | 2 | 4 | 6 | 8 | 10 | 12 | 16 | 18 | 20 | 22 | 28 | 30 | 32 | 36 | 40 | 42 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\phi^{-1}(n)=p^a$ | 2 | 3 | 5 | 7 | | 11 | 13 | 17 | 19 | | 23 | 29 | 31 | | 37 | 41 | 43 |
| | | $2^2$ | $2^3$ | $3^2$ | $2^4$ | | | $2^5$ | $3^3$ | $5^2$ | | | | $2^6$ | | | $7^2$ |

17. *Computation of* $\phi^{-1}(n)$. Let $m=\phi^{-1}(n)$. If $m=2^{a_0}p_1^{a_1}\cdots p_r^{a_r}$, then $n=\phi(2^{a_0})\cdot\phi(p_1^{a_1})\cdots\phi(p_r^{a_r})$. Now let

$$(1)\qquad \begin{cases} n_0=\phi(2^{a_0}), \text{ i. e., } 2^{a_0}=\phi^{-1}(n_0), \\ n_1=\phi(p_1^{a_1}), \quad " \quad p_1^{a_1}=\phi^{-1}(n_1), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ n_r=\phi(p_r^{a_r}), \quad " \quad p_r^{a_r}=\phi^{-1}(n_r). \end{cases}$$

Then $n=n_0n_1\cdots n_r$ and $\phi^{-1}(n)=\phi^{-1}(n_0)\cdot\phi^{-1}(n_1)\cdots\phi^{-1}(n_r)$.

Therefore, to compute the values of $\phi^{-1}(n)$ we first resolve $n$ in every possible way into even factors $n_0, n_1, \cdots, n_r$ (except that $n_0$ may be 1), such that the values of their inverse $\phi$-functions can be taken to be powers of distinct primes; then for every such resolution of $n$ we select those powers of distinct primes in every possible way and form their product. The products so formed

*SCHOUTE, *Mehrdimensionale Geometrie* (1902), vol. 1, art. 112.

are the required values of $\phi^{-1}(n)$. The work will be shortened somewhat by ignoring the case $n_0 = 1$ until the end and then multiplying every odd value of $\phi^{-1}(n)$ by 2 to form an additional even value.

### 18. *Table of values of* $\phi^{-1}(n)$.

*The finite periods of n-ary irreducible matrices.*

| Degree $= n$ | | Period $= \phi^{-1}(n)$ | Degree $= n$ | | Period $= \phi^{-1}(n)$ | Degree $= n$ | | Period $= \phi^{-1}(n)$ |
|---|---|---|---|---|---|---|---|---|
| $n$ | $= n_0 \dots n_r$ | $2^{a_0}p_1^{a_1}\dots p_r^{a_r} = m$ | $n$ | $= n_0 \dots n_r$ | $2^{a_0}p_1^{a_1}\dots p_r^{a_r} = m$ | $n$ | $= n_0 \dots n_r$ | $2^{a_0}p_1^{a_1}\dots p_r^{a_r} = m$ |
| 1 | $=1$ | $1 = 1$ | 10 | $=10$ | $11 = 11$ | 20 | $=20$ | $5^2 = 25$ |
| | | $2 = 2$ | | | $2\cdot 11 = 22$ | | | $2\cdot 5^2 = 50$ |
| 2 | $=2$ | $3 = 3$ | 12 | $=12$ | $13 = 13$ | | $=2\cdot 10$ | $3\cdot 11 = 33$ |
| | | $2\cdot 3 = 6$ | | | $2\cdot 13 = 26$ | | | $2\cdot 3\cdot 11 = 66$ |
| | | $2^2 = 4$ | | $=2\cdot 6$ | $3\cdot 7 = 21$ | | | $2^2\cdot 11 = 44$ |
| 4 | $=4$ | $5 = 5$ | | | $2\cdot 3\cdot 7 = 42$ | 22 | $=22$ | $23 = 23$ |
| | | $2\cdot 5 = 10$ | | | $2^2\cdot 7 = 28$ | | | $2\cdot 23 = 46$ |
| | | $2^3 = 8$ | | | $2^2\cdot 3^2 = 36$ | 24 | $=2\cdot 12$ | $3\cdot 13 = 39$ |
| | $=2\cdot 2$ | $2^2\cdot 3 = 12$ | 16 | $=16$ | $17 = 17$ | | | $2\cdot 3\cdot 13 = 78$ |
| 6 | $=6$ | $7 = 7$ | | | $2\cdot 17 = 34$ | | | $2^2\cdot 13 = 52$ |
| | | $2\cdot 7 = 14$ | | | $2^5 = 32$ | | $=4\cdot 6$ | $5\cdot 7 = 35$ |
| | | $3^2 = 9$ | | $=8\cdot 2$ | $2^4\cdot 3 = 48$ | | | $2\cdot 5\cdot 7 = 70$ |
| | | $2\cdot 3^2 = 18$ | | $=4\cdot 4$ | $2^3\cdot 5 = 40$ | | | $2^3\cdot 7 = 56$ |
| 8 | $=8$ | $2^4 = 16$ | | $=2\cdot 2\cdot 4$ | $2^2\cdot 3\cdot 5 = 60$ | | | $2^3\cdot 3^2 = 72$ |
| | $=2\cdot 4$ | $3\cdot 5 = 15$ | 18 | $=18$ | $19 = 19$ | | $=6\cdot 4$ | $3^2\cdot 5 = 45$ |
| | | $2\cdot 3\cdot 5 = 30$ | | | $2\cdot 19 = 38$ | | | $2\cdot 3^2\cdot 5 = 90$ |
| | | $2^2\cdot 5 = 20$ | | | $3^3 = 27$ | | $=2\cdot 2\cdot 6$ | $2^2\cdot 3\cdot 7 = 84$ |
| | $=4\cdot 2$ | $2^3\cdot 3 = 24$ | | | $2\cdot 3^3 = 54$ | | | |

19. *Computation of* $\psi^{-1}(n)$. Let $m = \psi^{-1}(n)$. If $m = 2^{a_0}p_1^{a_1}\cdots p_r^{a_r}$, where $a_0 > 1$, then $n = \phi(2^{a_0}) + \phi(p_1^{a_1}) + \cdots + \phi(p_r^{a_r})$. Defining $n_0, n_1, \cdots, n_r$, as before, by equations (1), we obtain the final equations $n = n_0 + n_1 + \cdots + n_r$, and $\psi^{-1}(n) = \phi^{-1}(n_0)\cdot\phi^{-1}(n_1)\cdots\phi^{-1}(n_r)$.

Therefore the computation of $\psi^{-1}(n)$ is strictly analogous to that of $\phi^{-1}(n)$. We first form all the possible partitions of $n$ into even parts $n_0, n_1, \cdots, n_r$, such that the values of their inverse $\phi$-functions can be taken to be powers of distinct primes; then for every such partition of $n$ we select those powers of distinct primes in every possible way and form their product. The products so formed, together with the doubles of those that are odd, are the required values of $\psi^{-1}(n)$.

## 20. *Table of values of* $\psi^{-1}(n)$.

*Those finite periods of n-ary matrices which are not also periods of* $(n-1)$*-ary matrices.*

| Degree $= n$ | | Period $= \psi^{-1}(n)$ | Degree $= n$ | | Period $= \psi^{-1}(n)$ | Degree $= n$ | | Period $= \psi^{-1}(n)$ |
|---|---|---|---|---|---|---|---|---|
| $n$ | $=n_0+\cdots+n_r$ | $2^{a_0}p_1^{a_1}\cdots p_r^{a_r}=m$ | $n$ | $=n_0+\cdots+n_r$ | $2^{a_0}p_1^{a_1}\cdots p_r^{a_r}=m$ | $n$ | $=n_0+\cdots+n_r$ | $2^{a_0}p_1^{a_1}\cdots p_r^{a_r}=m$ |
| 1 | $=1$ | $1=1$ | 8 | $=8$ | $2^4=16$ | 12 | $=12$ | $13=13$ |
| | | $2=2$ | | $=2+6$ | $3\cdot7=21$ | | | $21\cdot3=26$ |
| 2 | $=2$ | $3=3$ | | | $2\cdot3\cdot7=42$ | | $=2+10$ | $3\cdot11=33$ |
| | | $2\cdot3=6$ | | | $2^2\cdot7=28$ | | | $2\cdot3\cdot11=66$ |
| | | $2^2=4$ | | | $2^2\cdot3^2=36$ | | | $2^2\cdot11=44$ |
| 4 | $=4$ | $5=5$ | | $=4+4$ | $2^3\cdot5=40$ | | $=8+4$ | $2^4\cdot5=80$ |
| | | $2\cdot5=10$ | | $=2+2+4$ | $2^2\cdot3\cdot5=60$ | | $=6+6$ | $3^2\cdot7=63$ |
| | | $2^3=8$ | 10 | $=10$ | $11=11$ | | | $2\cdot3^2\cdot7=126$ |
| | $=2+2$ | $2^2\cdot3=12$ | | | $2\cdot11=22$ | | $=2+4+6$ | $3\cdot5\cdot7=105$ |
| 6 | $=6$ | $7=7$ | | $=8+2$ | $2^4\cdot3=48$ | | | $2\cdot3\cdot5\cdot7=210$ |
| | | $2\cdot7=14$ | | $=4+6$ | $5\cdot7=35$ | | | $2^2\cdot5\cdot7=140$ |
| | | $3^2=9$ | | | $2\cdot5\cdot7=70$ | | $=4+2+6$ | $2^3\cdot3\cdot7=168$ |
| | | $2\cdot3^2=18$ | | | $2^3\cdot7=56$ | | $=2+6+4$ | $2^2\cdot3^2\cdot5=180$ |
| | $=2+4$ | $3\cdot5=15$ | | | $2^3\cdot3^2=72$ | | | |
| | | $2\cdot3\cdot5=30$ | | $=6+4$ | $3^2\cdot5=45$ | | | |
| | | $2^2\cdot5=20$ | | | $2\cdot3^2\cdot5=90$ | | | |
| | $=4+2$ | $2^3\cdot3=24$ | | $=2+2+6$ | $2^2\cdot3\cdot7=84$ | | | |
| | | | | $=4+2+4$ | $2^3\cdot3\cdot5=120$ | | | |

### Canonical forms.

21. We wish to determine the canonical forms of all $n$-ary matrices of period $m$ in the domain (1). The canonical form of a reducible matrix must be a reduced form whose irreducible components are all in their canonical forms, but the latter, i. e., the canonical forms of irreducible matrices, can be chosen in a variety of ways, the desideratum being to have as large a number of zero elements as possible. One effective method is to use the lemma of § 2. Another method, which to some extent agrees with the former, and which makes every canonical form, at bottom, a permutation, will now be outlined; the canonical form of an irreducible matrix $L$ will be determined for different values of its period $m$.

22. $m=p$, a prime. In this case $L$ is of degree $p-1$. Assume $p$ letters $X_1, \cdots, X_p$, connected by the symmetrical relation $X_1+\cdots+X_p=0$ (geometrically, $p$ coördinates, of which $p-1$ are independent). Any cyclical permutation on all the letters is of period $p$ and is equivalent to a linear substitution on the first $p-1$ of them; in particular, the permutation $(X_1\cdots X_p)$ is equivalent to the linear substitution

$$X_1'=X_2,\ X_2'=X_3,\ \cdots,\ X_{p-1}'=-X_1-\cdots-X_{p-1},$$

and therefore to the matrix

$$L' = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -1 & -1 & -1 & \cdots & -1 \end{bmatrix},$$

of degree $p-1$.* *$L'$ will be taken to be the canonical form of $L$.* For example, if $p=2$, $L'=(-1)$; if $p=3$, $L'=\left(\begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix}\right)$.

23. $m=p^a$ ($p$ a prime). In this case $L$ is of degree $p^{a-1}(p-1)$. Assume $p^a$ letters $X_1, \cdots, X_{p^a}$ connected by the $p^{a-1}$ symmetrical relations

$$\sum_{j=0}^{p-1} X_{i+jp^{a-1}} = 0 \qquad (i=1, \cdots, p^{a-1}) \tag{2}$$

(geometrically, $p^a$ coördinates, of which $p^{a-1}(p-1)$ are independent). Take any cyclical permutation on all the letters which leaves the relations (2) invariant and is therefore imprimitive, having the systems of letters entering into those relations as imprimitive systems. Such a permutation is of period $p^a$ and is equivalent to a linear substitution on the $p^{a-1}(p-1)$ letters $X_1, \cdots, X_{p^{a-1}(p-1)}$, which are obtained by omitting the last letter of each imprimitive system.

In particular, the permutation $(X_1 \cdots X_{p^a})$ is equivalent to the linear substitution

$$X_1' = X_2,\ X_2' = X_3,\ \cdots,\ X_{p^{a-1}(p-1)}' = -\sum_{j=0}^{p-2} X_{1+jp^{a-1}}.$$

*The corresponding matrix will be taken to be the canonical form of every irreducible matrix of period $p^a$.*

For example, if $m=3^2$, we have nine letters $X_1, \cdots, X_9$ connected by the three relations

$$X_1 + X_4 + X_7 = X_2 + X_5 + X_8 = X_3 + X_6 + X_9 = 0;$$

the permutation $(X_1 \cdots X_9)$ is equivalent to the linear substitution

$$X_1' = X_2,\ X_2' = X_3,\ X_3' = X_4,\ X_4' = X_5,\ X_5' = X_6,\ X_6' = -X_1 - X_4$$

* This method, for the case $m=p$, was given, in substance, in my paper in the Bulletin of the American Mathematical Society (1907), vol. 13, p. 343, § 14. Cf. also KLEIN's $n$-ary group of order $(n+1)!$ (KLEIN, Mathematische Annalen, vol. 4 (1871), pp. 346–358; MOORE, American Journal of Mathematics (1900), vol. 22, pp. 336–342.)

and therefore to the canonical matrix

$$\begin{vmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 & 0 & 0 \end{vmatrix}.$$

Again, if $m = 2^2$, we have four letters $X_1$, $X_2$, $X_3$, $X_4$, connected by the two relations, $X_1 + X_3 = X_2 + X_4 = 0$; the resulting canonical form of an irreducible matrix of period 4 is $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.

The method of § 22 is evidently a special case of this, for $a = 1$. In these two sections the result agrees with that obtained by means of the lemma of § 2.

24. $m = 2p^a$ ($p$ an odd prime). In this case $L$ is again of degree $p^{a-1}(p-1)$. Its canonical form is easily obtained from that of an irreducible matrix of period $p^a$ by changing the signs of all its elements. For example, if $m = 2 \cdot 3$, the canonical form is $\left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)$.

25. $m = p_1^{a_1} \cdots p_r^{a_r}$ ($p_1, \cdots, p_r$ being distinct primes). In this general case $L$ is of degree $\phi(m) = p_1^{a_1-1}(p_1 - 1) \cdots p_r^{a_r-1}(p_r - 1)$. Consider the $m$ letters

$$X_{i_1, \ldots, i_r} \quad (i_1 = 1, \cdots, p_1^{a_1}; \cdots; i_r = 1, \cdots, p_r^{a_r}). \tag{3}$$

Suppose them connected by the sets of symmetrical relations

$$\begin{cases} \sum_{j=0}^{p_1-1} X_{i_1 + jp_1^{a_1-1}, i_2, \ldots, i_r} = 0 & (i_1 = 1, \cdots, p_1^{a_1-1}; i_2 = 1, \cdots, p_2^{a_2}; \cdots; i_r = 1, \cdots, p_r^{a_r}), \\ \cdots\cdots\cdots\

and is imprimitive, having the systems of letters entering into those relations as imprimitive systems. It is evidently equivalent to a linear substitution on the $\phi(m)$ letters (5), and therefore to a matrix of degree $(m)$.

In order to obtain a simple canonical form for the matrix, it will be convenient to extend the range of values of the suffixes $i_1, \cdots, i_r$ to include all integers, and to define every new letter $X_{i_1, \ldots, i_r}$ so obtained as equal to the old letter whose suffixes are the least positive residues of the suffixes of the new letter with respect to the moduli $p_1^{a_1}, \cdots, p_r^{a_r}$. More precisely, we put $X_{i_1, \ldots, i_r} = X_{i_1', \ldots, i_r'}$, if, and only if, $i_1 \equiv i_1'$, mod $p_1^{a_1}, \cdots$, and $i_r \equiv i_r'$, mod $p_r^{a_r}$. Then the letters

$$X_{i, i, \ldots, i} \qquad (i = 1, 2, \cdots, m)$$

are all distinct. For if $X_{i, i, \ldots, i} = X_{j, j, \ldots, j}$, then $i \equiv j$, mod $p_1^{a_1}$, mod $p_2^{a_2}, \cdots$, and mod $p_r^{a_r}$; and therefore $i \equiv j$, mod $m$. So, being $m$ in number, they are simply the letters (3) arranged in a particular order.

It is now clear that the permutation $(X_{1, 1, \ldots, 1} \cdots X_{m, m, \ldots, m})$, of period $m$, has the required imprimitive systems, and is therefore equivalent to the following linear substitution on the $\phi(m)$ letters (5):

$$X'_{i_1, \ldots, i_r} = X_{i_1+1, \ldots, i_r+1} \quad [i_1 = 1, \cdots, p_1^{a_1-1}(p_1-1); \cdots; i_r = 1, \cdots, p_r^{a_r-1}(p_r-1)],$$

in which it is understood that every supernumerary letter is to be expressed in terms of the letters (5) by means of the relations (4). *The resulting matrix will be taken to be the canonical form of every irreducible matrix of period m.*

The §§ 22–24 are evidently special cases of this section.

26. Examples. (1) If $m = 3 \cdot 5$, so that $\phi(m) = 2 \cdot 4$, we have the 15 letters

$$X_{11}, X_{12}, X_{13}, X_{14}, X_{15},$$

$$X_{21}, X_{22}, X_{23}, X_{24}, X_{25},$$

$$X_{31}, X_{32}, X_{33}, X_{34}, X_{35},$$

conditioned by the vanishing of the sum of the letters in each row and in each column. The permutation $(X_{11} \cdots X_{1515})$ becomes the octonary linear substitution $X'_{ij} = X_{i+1, j+1}$ $(i = 1, 2; j = 1, 2, 3, 4)$, which may be written

$$X'_{1j} = X_{2, j+1} \qquad (j = 1, 2, 3), \qquad X'_{14} = -\sum_{j=1}^{4} X_{2j},$$

$$X'_{2j} = -\sum_{i=1}^{2} X_{i, j+1} \quad (j = 1, 2, 3), \qquad X'_{24} = \sum_{i=1, 2}^{j=1, 2, 3, 4} X_{ij}.$$

Therefore the canonical form of an irreducible matrix of period 15 is

$$\begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{vmatrix}.$$

(2) If $m = 2^2 \cdot 3$, so that $\phi(m) = 2 \cdot 2$, we have the twelve letters $X_{ij}$ $(i = 1, 2, 3;\ j = 1, 2, 3, 4)$, connected by the relations

$$X_{i1} + X_{i3} = X_{i2} + X_{i4} = 0 \ (i = 1, 2, 3), \quad X_{1j} + X_{2j} + X_{3j} = 0 \ (j = 1, 2, 3, 4).$$

The permutation $(X_{11} \cdots X_{12\,12})$ becomes the quaternary linear substitution $X'_{ij} = X_{i+1,j+1}$ $(i, j = 1, 2)$, which may be written

$$X'_{11} = X_{22},\ X'_{12} = -X_{21},\ X'_{21} = -X_{12} - X_{22},\ X'_{22} = X_{11} + X_{21}.$$

Therefore the canonical form is

$$\begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 1 & 0 & 1 & 0 \end{vmatrix}.$$

27. We are now prepared to find the canonical forms of all $n$-ary matrices of period $m$. Theorem 4 shows that we merely have to *select in all possible ways integers* $m_1, \cdots, m_s$ *such that their L. C. M. is* $m$, *and* $\phi(m_1) + \cdots + \phi(m_s) = n$. *The number of canonical forms is equal to the number of ways of selecting these integers. The canonical form corresponding to any given selection is obtained by taking irreducible components of periods* $m_1, \cdots, m_s$ *and of degrees* $\phi(m_1), \cdots, \phi(m_s)$, *respectively, each in its canonical form as defined above, and combining them into a single reduced matrix.*

Example. If $n = 3$ and $m = 6$, we find three pairs of values of $m_1$ and $m_2$, namely 3 and 2, 6 and 1, 6 and 2, and therefore three canonical forms of ternary matrices of period 6, namely,

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}.$$

It will be noticed that every canonical form, as we have defined it, is a matrix whose elements are equal to 0, 1, or $-1$ and whose determinant is equal to $\pm 1$.

## Results for matrices of low degree.

28. The results of the foregoing theory may be applied to matrices of a few of the lowest degrees and summarized as follows:

$n = 2$. Apart from the trivial case $m = 1$, *the only finite periods of binary matrices with rational elements are* 2, 3, 6, *and* 4. The matrices of period 2 are all reducible; their canonical forms are $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, whose characteristic determinant is $(\lambda+1)^2$, and $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, whose characteristic determinant is $(\lambda-1)(\lambda+1)$. The matrices of periods 3, 6, and 4 are all irreducible; their canonical forms and characteristic determinants are $\left(\begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix}\right)$ and $\lambda^2+\lambda+1$, $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$ and $\lambda^2-\lambda+1$, $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $\lambda^2+1$, respectively.

$n = 3$. *In the field* (1) *ternary matrices have the same finite periods as binary matrices, and all ternary matrices of finite period are reducible.* Those of period 2 have three canonical forms, namely,

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

whose characteristic determinants are $(\lambda+1)^3$, $(\lambda-1)(\lambda+1)^2$, and $(\lambda-1)^2(\lambda+1)$, respectively. Those of period 3 can all be transformed into the single canonical form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix},$$

whose characteristic determinant is $(\lambda-1)(\lambda^2+\lambda+1)$. Those of period 4 have the two canonical forms

$$\begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

whose characteristic determinants are $(\lambda \mp 1)(\lambda^2+1)$. Finally, those of period 6 have the three canonical forms mentioned in § 27, the corresponding characteristic determinants being $(\lambda+1)(\lambda^2+\lambda+1)$ and $(\lambda \mp 1)(\lambda^2-\lambda+1)$, respectively.*

$n = 4$. *Quaternary matrices have the periods* 5, 10, 8, *and* 12, *in addition to those of binary matrices.* All quaternary matrices of periods 5, 10, and 8,

* Contrast these nine canonical forms of ternary matrices, all of which are reduced, with the eight canonical forms, four reduced and four irreducible, obtained by POINCARÉ (l. c., p. 448).

and some of period 12, are irreducible; all those of periods 2, 3, 6, and 4, and some of period 12, are reducible. Those of period 2 have four canonical forms, which require no further mention. Those of period 3 have two canonical forms, whose characteristic determinants are $(\lambda-1)^2(\lambda^2+\lambda+1)$ and $(\lambda^2+\lambda+1)^2$, respectively. Those of period 6 have seven canonical forms, and those of period 4 have four canonical forms.

Those whose periods are 5, 10, and 8 have the canonical forms and characteristic determinants

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 \end{bmatrix} \quad \text{and} \quad \lambda^4+\lambda^3+\lambda^2+\lambda+1,$$

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \lambda^4-\lambda^3+\lambda^2-\lambda+1,$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \lambda^4+1,$$

respectively. Those matrices of period 12 which are irreducible have the canonical form

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

and the characteristic determinant $\lambda^4-\lambda^2+1$, while those which are reducible have the two canonical forms

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

whose characteristic determinants are $(\lambda^2+1)(\lambda^2+\lambda+1)$ and $(\lambda^2+1)(\lambda^2-\lambda+1)$, respectively.

It will be noticed that every canonical form, as we have defined it, is a matrix whose elements are equal to 0, 1, or $-1$ and whose determinant is equal to $\pm 1$.

## Results for matrices of low degree.

28. The results of the foregoing theory may be applied to matrices of a few of the lowest degrees and summarized as follows:

$n = 2$. Apart from the trivial case $m = 1$, *the only finite periods of binary matrices with rational elements are* 2, 3, 6, *and* 4. The matrices of period 2 are all reducible; their canonical forms are $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, whose characteristic determinant is $(\lambda + 1)^2$, and $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, whose characteristic determinant is $(\lambda-1)(\lambda+1)$. The matrices of periods 3, 6, and 4 are all irreducible; their canonical forms and characteristic determinants are $\left(\begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix}\right)$ and $\lambda^2 + \lambda + 1$, $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$ and $\lambda^2-\lambda+1$, $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $\lambda^2 + 1$, respectively.

$n = 3$. *In the field* (1) *ternary matrices have the same finite periods as binary matrices, and all ternary matrices of finite period are reducible.* Those of period 2 have three canonical forms, namely,

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

whose characteristic determinants are $(\lambda+1)^3$, $(\lambda-1)(\lambda+1)^2$, and $(\lambda-1)^2(\lambda+1)$, respectively. Those of period 3 can all be transformed into the single canonical form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix},$$

whose characteristic determinant is $(\lambda - 1)(\lambda^2 + \lambda + 1)$. Those of period 4 have the two canonical forms

$$\begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

whose characteristic determinants are $(\lambda \mp 1)(\lambda^2 + 1)$. Finally, those of period 6 have the three canonical forms mentioned in § 27, the corresponding characteristic determinants being $(\lambda + 1)(\lambda^2 + \lambda + 1)$ and $(\lambda \mp 1)(\lambda^2-\lambda+1)$, respectively.*

$n = 4$. *Quaternary matrices have the periods* 5, 10, 8, *and* 12, *in addition to those of binary matrices.* All quaternary matrices of periods 5, 10, and 8,

* Contrast these nine canonical forms of ternary matrices, all of which are reduced, with the eight canonical forms, four reduced and four irreducible, obtained by POINCARÉ (l. c., p. 448).

and some of period 12, are irreducible; all those of periods 2, 3, 6, and 4, and some of period 12, are reducible. Those of period 2 have four canonical forms, which require no further mention. Those of period 3 have two canonical forms, whose characteristic determinants are $(\lambda-1)^2(\lambda^2+\lambda+1)$ and $(\lambda^2+\lambda+1)^2$, respectively. Those of period 6 have seven canonical forms, and those of period 4 have four canonical forms.

Those whose periods are 5, 10, and 8 have the canonical forms and characteristic determinants

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 \end{bmatrix} \quad \text{and} \quad \lambda^4+\lambda^3+\lambda^2+\lambda+1,$$

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \lambda^4-\lambda^3+\lambda^2-\lambda+1,$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \lambda^4+1,$$

respectively. Those matrices of period 12 which are irreducible have the canonical form

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

and the characteristic determinant $\lambda^4-\lambda^2+1$, while those which are reducible have the two canonical forms

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

whose characteristic determinants are $(\lambda^2+1)(\lambda^2+\lambda+1)$ and $(\lambda^2+1)(\lambda^2-\lambda+1)$, respectively.

$n = 5$. Quinary matrices have the same finite periods as quaternary matrices; all quinary matrices of finite period are reducible.

$n = 6$. Senary matrices have eight additional periods, viz., 7, 14, 9, 18, 15, 30, 20, and 24, making seventeen periods in all. Those whose periods are 7, 14, 9, and 18, are irreducible; those whose periods are 15, 30, 20, and 24 are reducible, and their reduced forms are each made up of two components, one binary and one quaternary. There is a single canonical form corresponding to each of the periods 7, 14, 9, 18, 15, 20, and 24, while there are three canonical forms corresponding to the period 30.

$n = 7$. Septenary matrices have the same finite periods as senary matrices; all septenary matrices of finite period are reducible.

$n = 8$. Octonary matrices have seven additional periods, 16, 21, 42, 28, 36, 40, and 60. Here for the first time we meet with irreducible matrices of periods 15, 30, 20, and 24. Besides these, there are also reducible octonary matrices of periods 15, 30, 20, and 24. For example, the irreducible matrices of period 20 have the characteristic determinant $\lambda^8 - \lambda^6 + \lambda^4 - \lambda^2 + 1$, while the reducible matrices of period 20 have eight different characteristic determinants, one of which is $(\lambda^4 + \lambda^3 + \lambda^2 + \lambda + 1)(\lambda^2 + 1)^2$.

This classification can easily be extended to cover matrices of any given degree whatever.*

CORNELL UNIVERSITY,
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* I wish to acknowledge my obligation to Professor FITE for valuable assistance in correcting errors and omissions in this paper.

# ON HYPERCOMPLEX NUMBER SYSTEMS BELONGING TO AN ARBITRARY DOMAIN OF RATIONALITY*

BY

R. B. ALLEN

In this paper I shall consider only number systems with units $e_1, e_2, \cdots, e_n$, whose constants of multiplication, $\gamma_{ijk}$, lie in $R$, a domain of rationality determined by an arbitrary aggregation of scalars which form a closed system with respect to addition, subtraction, multiplication and division; such systems are said to *belong to* $R$. The units $e_1, e_2, \cdots, e_n$ will be so chosen that there shall exist between them no linear relation with coefficients in $R$. In general I shall consider only numbers

$$A = \sum_{i=1}^{n} a_i e_i,$$

of a number system belonging to $R$, for which the $a$'s lie in $R$; all such numbers form a closed system with respect to the operations of multiplication, addition and subtraction, and the totality of such numbers is said to constitute the *hypercomplex domain* $\Re(R, e_i)$ in which the numbers $A$ lie. Thus if $A$ and $B$ lie in $\Re(R, e_i)$ so also does $A \pm B$ and also $A \cdot B$, since $R$ contains the constants of multiplication of the system. Moreover, if $\rho$ lies in $R$, $\rho A$ lies in $\Re(R, e_i)$. The introduction of the conception of a domain of rationality necessitates a revision of certain fundamental definitions as usually accepted, such as of reducibility and of equivalence.

A transformation,

$$(e_1', e_2', \cdots, e_n') = T(e_1, e_2, \cdots, e_n),$$

of the units is said to be *rational with respect to* $R$ if the coefficients of $T$ lie in $R$. The transformation $T$ is always assumed to be linear and with determinant not zero.

Let $T$ be rational with respect to $R$. Then,

(i) The $e'$s lie in $\Re(R, e_i)$;

(ii) The hypercomplex domains $\Re(R, e_i)$ and $\Re(R, e_i')$ are identical;

(iii) The system $e_1', e_2', \cdots, e_n'$ belongs to $R$.

Two number systems are *equivalent with respect to* $R$ if one can be trans-

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formed into the other by a transformation rational with respect to $R$; otherwise *inequivalent with respect to* $R$.

A number system is *reducible with respect to* $R$ if it is equivalent with respect to $R$ to a system which can be separated into two mutually exclusive subsystems such that the product of each unit of one subsystem by each unit of the other subsystem, both as a prefactor and as a postfactor, is zero; such subsystems are said to be *mutually nilfactorial.*

The general theory of hypercomplex number systems, as usually treated, coincides with the present theory when $R$ contains all real and imaginary scalars, and the theory of real hypercomplex number systems coincides with the present theory when $R$ is restricted to real scalars. The conception which is the basis of the present paper was set forth by H. TABER* in these Transactions, vol. 5 (1904), pp. 509–548. In what follows, unless otherwise explicitly stated, $A$, $A_i$, $B$, etc., will denote hypercomplex numbers lying in $\mathfrak{R}(R, e_i)$ and $a$, $a_i$, $b$, etc., will denote scalars lying in $R$. Moreover, unless the contrary is stated, any transformation of the units will be assumed to be linear with non-zero determinant and rational with respect to $R$. In general I shall employ the nomenclature of B. PEIRCE.†

## § 1. *The fundamental equation.*

For any given number

$$A = \sum_{i=1}^{n} a_i e_i$$

there is a smallest positive integer $\nu$ for which $A$, $A^2$, $A^3$, $\cdots$, $A^\nu$ are linearly related in $\mathfrak{R}(R, e_i)$; this relation

$$\Omega(A) \equiv A^\nu + p_1 A^{\nu-1} + \cdots + p_{\nu-1} A = 0 \qquad (p\text{'s in } R)$$

is called the *fundamental equation of* $A$. If

$$\phi(A) \equiv A^{\nu'} + p_1' A^{\nu'-1} + \cdots + p_{\nu'-1}' A = 0 \qquad (p'\text{'s in } R),$$

then‡ for any scalar $\lambda$,

$$\phi(\lambda) \equiv \sum_{p=0}^{\nu'-\nu} c_p \lambda^p \Omega(\lambda) \qquad (c\text{'s in } R).$$

Let

$$\Omega(\lambda) \equiv \lambda^m \cdot [\Omega_1(\lambda)]^{m_1} \cdot [\Omega_2(\lambda)]^{m_2} \cdot \cdots \cdot [\Omega_s(\lambda)]^{m_s}$$

where $\Omega_1(\lambda)$, $\Omega_2(\lambda)$, $\cdots$, $\Omega_s(\lambda)$ of orders $r_1$, $r_2$, $\cdots$, $r_s$ are the distinct factors of

* This paper will be cited under $T_1$. A second paper by H. TABER, that will soon appear and that I have consulted, will be cited under $T_2$.

† American Journal of Mathematics, vol. 4 (1882), p. 97. Cf. H. E. HAWKES, these Transactions, vol. 3 (1902), p. 312; and American Journal of Mathematics, vol. 24 (1902), p. 87.

‡ TABER, $T_1$, p. 513.

$\Omega(\lambda)$, irreducible with respect to $R$. The greatest value $s$ may have for any number $A$ in the system will be denoted by $\bar{s}$. Any transformation $T$, rational in $R$, of the units, leaves $\bar{s}$ unaltered.* Let $m$ be the greatest value $\mathfrak{m}$ may have for any number $A$ for which $s = \bar{s}$; for $i = 1, 2, \cdots, \bar{s}$, severally, let $r_i$ be the greatest value $\mathfrak{r}_i$ may have for any number $A$ for which $s = \bar{s}$; and for $i = 1, 2, \cdots, \bar{s}$, severally, let $m_i$ be the greatest value $\mathfrak{m}_i$ may have for any number $A$ for which $s = \bar{s}$ and $\mathfrak{r}_i = r_i$.

Among the numbers $A$ for which $s = \bar{s}$ one or more can be found for which simultaneously †

$$\mathfrak{m} = m, \qquad \mathfrak{r}_i = r_i, \qquad \mathfrak{m}_i = m_i \qquad (i = 1, 2, \cdots, \bar{s}).$$

I shall assume that the number $A$ has been chosen to fulfill these conditions. Then, corresponding to the irreducible factors $\Omega_i(\lambda)$ of $\Omega(\lambda)$ $(i = 1, 2, \cdots, \bar{s})$, there are $\bar{s}$ independent numbers $I_i = f_i(A)$, lying in $\mathfrak{N}(R, e_i)$, the $f_i$ being polynomials in $A$ with coefficients rational in $R$, such that

$$I_i I_i = I_i, \qquad I_i I_j = 0 \qquad (i, j = 1, 2, \cdots, \bar{s};\ i \neq j).$$

With respect to these $\bar{s}$ idempotent numbers, $I_1, I_2, \cdots, I_{\bar{s}}$, the system may be *regularized*, that is, the units may be so chosen, by a transformation rational in $R$, that each will fall into one of the following groups of units:

$$\begin{array}{llllll}
\overline{11}, & \overline{12}, & \cdots, & \overline{1\bar{s}}, & \overline{10}, \\
\overline{21}, & \overline{22}, & \cdots, & \overline{2\bar{s}}, & \overline{20}, \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\overline{\bar{s}1}, & \overline{\bar{s}2}, & \cdots, & \overline{\bar{s}\bar{s}}, & \overline{\bar{s}0}, \\
\overline{01}, & \overline{02}, & \cdots, & \overline{0\bar{s}}, & \overline{00},
\end{array}$$

such that, when any number of the group $\overline{ij}$, lying in $\mathfrak{N}(R, e_i)$, is represented by $(ij)$, $(ij)_1$, $(ij)_2$, etc., the following table gives the group of the product of numbers belonging to the several groups:

| | $(ii)_1$ | $(ij)_1$ | $(ji)_1$ | $(jj)_1$ |
|---|---|---|---|---|
| $(ii)$ | $(ii)_2$ | $(ij)_2$ | 0 | 0 |
| $(ij)$ | 0 | 0 | $(ii)_2$ | $(ij)_2$ |
| $(ji)$ | $(ji)_2$ | $(jj)_2$ | 0 | 0 |
| $(jj)$ | 0 | 0 | $(ji)_2$ | $(jj)_2$ |

Moreover, for $1 \leqq i \leqq \bar{s}$:

*TABER, $T_2$.

†TABER, $T_2$.

(1) The idempotent number $I_i$ belongs to the groups $\bar{i}\bar{i}$ the units of which constitute a subsystem; $I_i$ is the modulus of this subsystem, and is the only idempotent number in it lying in $\Re(R, e_i)$.

(2) The nilpotent numbers of the group $\bar{i}\bar{i}$ lying in $\Re(R, e_i)$, constitute an invariant subsystem of the system $\bar{i}\bar{i}$, that is, the product of any such number and any number of the group $\bar{i}\bar{i}$, lying in $\Re(R, e_i)$, both as prefactor and as postfactor, is a nilpotent number of the group $\bar{i}\bar{i}$, lying in $\Re(R, e_i)$.

(3) Among the numbers of the group $\bar{i}\bar{i}$ are three numbers, $A_i$, $\bar{A}_i$ and $A_i'$, polynomials in $A$ with coefficients rational in $R$, whose fundamental equations are, respectively,

$$\omega_i(A_i) = 0, \qquad \bar{\omega}_i(\bar{A}_i) = 0, \qquad \omega_i'(A_i') = 0,$$

where

$$\omega_i(\lambda) \equiv \lambda[\Omega_i(\lambda)]^{m_i}, \quad \bar{\omega}_i(\lambda) \equiv \lambda[\Omega_i(\lambda)], \qquad \omega_i'(\lambda) \equiv \lambda^{m_i}.$$

The numbers $I_i$, $\bar{A}_i$, $\bar{A}_i^2$, $\cdots$, $\bar{A}_i^{r_i-1}$ are independent and

$$B \equiv c_0 I_i + c_1 \bar{A}_i + c_2 \bar{A}_i^2 + \cdots + c_{r_i-1} \bar{A}_i^{r_i-1}$$

has a reciprocal with respect to $I_i$ unless each $c_i = 0$.

(4) $I_i(ij) = (ij)$, $(ji)I_i = (ji)$ $(j = 0, 1, 2, \cdots, \bar{s})$.

(5) The units of $\overline{00}$ form a nilpotent system, in which there is a number $A_0$, a polynomial in $A$ with coefficients rational in $R$, such that its fundamental equation is

$$\omega_0(A_0) = 0,$$

where

$$\omega_0(\lambda) \equiv \lambda^m.$$

Finally we may substitute for $A$ the number $A_0 + A_1 + A_2 + \cdots + A_{\bar{s}}$, for which $\mathfrak{m} = m$, $\mathfrak{r}_i = r_i$, $\mathfrak{m}_i = m_i$ $(i = 1, 2, \cdots, \bar{s})$. Then, for $i = 0, 1, 2, \cdots, \bar{s}$, $A_i$ is a polynomial in this new number $\sum A_i$ (which from this point on we shall call $A$), with coefficients rational in $R$.*

Within $\Re(R, e_i)$ a nilpotent system belonging to $R$ is of *order* $p$ if the $p$th power of every number of the system is zero, while the $(p-1)$th power of some number of the system is not zero.

THEOREM I. *For $1 \leqq i \leqq \bar{s}$, the nilpotent numbers of the system $\bar{i}\bar{i}$ commutative with $\bar{A}_i$ form a nilpotent subsystem whose order is $m_i$. The units of the system $\overline{00}$ form a nilpotent subsystem whose order is $m$.*

The nilpotent numbers of the group $\bar{i}\bar{i}$ constitute a subsystem, and therefore the nilpotent numbers of $\bar{i}\bar{i}$ commutative with $\bar{A}_i$ form a nilpotent subsystem. Suppose there were in $\bar{i}\bar{i}$ a number $N_i$ such that

$$N_i \bar{A}_i = \bar{A}_i N_i, \qquad N_i^{m_i+q} \neq 0, \qquad N_i^{m_i+q+1} = 0.$$

* For the proof of these theorems see TABER, $T_2$.

Now the numbers

$$\bar{A}_i,\ \bar{A}_i^2,\ \cdots,\ \bar{A}_i^{r_i-1};\qquad \bar{A}_iN_i,\ \bar{A}_i^2N,\ \cdots,\ \bar{A}_i^{r_i-1}N_i;\qquad \cdots;\qquad \cdots;$$
$$\bar{A}_iN_i^{m_i+q},\ \bar{A}_i^2N_i^{m_i+q},\ \cdots,\ \bar{A}_i^{r_i-1}N_i^{m_i+q}$$

are independent. For, otherwise, on multiplying by a proper power of $N_i$, we would get a relation of the form

$$(c_1\bar{A}_i + c_2\bar{A}_i^2 + \cdots + c_{r_i-1}\bar{A}_i^{r_i-1})N_i^{m_i+q} = 0$$

where the $c$'s are not all zero; but in that event $\sum_{j=1}^{r_i-1} c_j\bar{A}_i^j$ has a reciprocal with respect to $I_i$ [see § 1 (1)], and therefore $N_i^{m_i+q} = 0$, which is contrary to hypothesis. Let

$$B_i = \bar{A}_i + N_i,$$

and let

$$\phi(B_i) \equiv B_i^{\mu} + b_2B_i^{\mu-1} + \cdots + b_{\mu}B_i = 0$$

be the fundamental equation of $B_i$. Since $\bar{A}_i$ and $N_i$ are commutative

$$\phi(B_i) = \phi(\bar{A}_i + N_i) = \phi(\bar{A}_i) + \phi'(\bar{A}_i)\cdot N_i + \frac{\phi''(\bar{A}_i)\cdot N_i^2}{2!}$$
$$+ \cdots + \frac{\phi^{(m_i+q)}(\bar{A}_i)\cdot N_i^{m_i+q}}{(m_i+q)!} = 0,$$

where $\phi^p(\lambda) \equiv d^p\phi(\lambda)/d\lambda^p$. Therefore

$$\phi(\bar{A}_i) = \phi'(\bar{A}_i) = \cdots = \phi^{(m_i+q)}(\bar{A}_i) = 0;$$

for otherwise there would be a relation among the $\bar{A}_i^kN_i^j$'s. Whence it follows (see § 1, first part) that

$$\phi(\lambda) \equiv \lambda\,[\Omega_i(\lambda)]^{m_i+q+1}.$$

Let

$$B = A_0 + A_1 + \cdots + A_{i-1} + B_i + A_{i+1} + \cdots + A_{\bar{s}}.$$

Then, since $A_j$ $(j = 0, 1, 2, \cdots, \bar{s})$ is a number of the group $\bar{\bar{j}}$, we have

$$B^p = A_0^p + A_1^p + \cdots + A_{i-1}^p + B_i^p + A_{i+1}^p + \cdots + A_{\bar{s}}^p,$$

and if $\psi(B) = 0$ is the fundamental equation of $B$,

$$\psi(A_0) = \psi(A_1) = \cdots = \psi(A_{i-1}) = \psi(B_i) = \psi(A_{i+1}) = \cdots = \psi(A_{\bar{s}}) = 0.$$

Therefore

$$\psi(\lambda) \equiv \lambda^m[\Omega_1(\lambda)]^{m_1}\cdots[\Omega_{i-1}(\lambda)]^{m_{i-1}}\cdot[\Omega_i(\lambda)]^{m_i+q+1}[\Omega_{i+1}(\lambda)]^{m_{i+1}}\cdots[\Omega_{\bar{s}}(\lambda)]^{m_{\bar{s}}},$$

and thus there is a number $B$ for which

$$s = \bar{s},\ \ \mathfrak{r}_j = r_j\ (j = 1, 2, \cdots, \bar{s}),\ \ \mathfrak{m}_k = m_k\ (k = 1, 2, \cdots, i-1, i+1, \cdots, \bar{s}),$$

and $\mathfrak{m}_i = m_i + q + 1 > m_i$, which is contrary to hypothesis. Consequently no such number as $N_i$ exists in $\bar{\bar{i}}$.

But by (3) there is a number $A_i'$, in $\bar{i}\bar{i}$, such that $A_i'^{m_i-1} \neq 0$, $A_i'^{m_i} = 0$, and such that $\bar{A}_i$ and $A_i'$ are commutative, both being polynomials in $A$. Hence follows the first part of the theorem.

Suppose there were a number $N_0$ in $\overline{00}$ such that

$$N_0^{m+q} \neq 0, \qquad N_0^{m+q+1} = 0.$$

Let $B = N_0 + A_1 + A_2 + \cdots + A_{\bar{s}}$, and let its fundamental equation be $\phi(B) = 0$. Then, since $N_0, A_1, A_2, \cdots, A_{\bar{s}}$ are mutually nilfactorial,

$$\phi(B) = \phi(N_0) + \phi(A_1) + \phi(A_2) + \cdots + \phi(A_{\bar{s}}) = 0.$$

Hence follows

$$\phi(N_0) = \phi(A_1) = \phi(A_2) = \cdots = \phi(A_{\bar{s}}) = 0,$$

and therefore

$$\phi(\lambda) \equiv \lambda^{m+q+1}[\Omega_1(\lambda)]^{m_1} \cdot [\Omega_2(\lambda)]^{m_2} \cdot \cdots \cdot [\Omega_{\bar{s}}(\lambda)]^{m_{\bar{s}}}.$$

and thus there is a number, viz., $B$, for which

$$s = \bar{s}, \qquad r_i = r_i, \qquad m_i = m_i \ (i = 1, 2, \cdots, \bar{s}), \qquad m = m + q + 1 > m,$$

which is contrary to hypothesis. Hence no such number as $N_0$ exists in $\overline{00}$.

There is, however, by (5), a number $A_0$ in $\overline{00}$ such that

$$A_0^{m-1} \neq 0, \qquad A_0^m = 0.$$

Hence follows the second part of the theorem.

## § 2. *Reducibility.*

THEOREM II. *The number of units in the group* $\bar{i}\bar{i}$ $(i = 1, 2, \cdots \bar{s}, )$ *is a multiple of* $r_i$, *and is not less than* $r_i m_i$.

Let $B_i$ be any number in $\bar{i}\bar{i}$. Then

$$B_i, \bar{A}_i B_i, \bar{A}_i^2 B_i, \cdots, \bar{A}_i^{r_i-1} B_i$$

are independent numbers in $\bar{i}\bar{i}$ and may be used as $r_i$ of the units. For, if

$$\sum_{p=0}^{r_i-1} c_p \bar{A}_i^p B_i = 0,$$

where the $c$'s are not all zero, we get, on multiplying by the reciprocal, with respect to $I_i$, of $\sum_{p=0}^{r_i-1} c_p \bar{A}_i^p$,

$$B_i = 0,$$

which is contrary to hypothesis. If there is a number $B_{ii}'$, in $\bar{i}\bar{i}$, independent of

$$B_i, \bar{A}_i B_i, \bar{A}_i^2 B_i, \cdots, \bar{A}_i^{r_i-1} B_i$$

then

$$B_i, \bar{A}_i B_i, \bar{A}_i^2 B_i, \cdots, \bar{A}^{r_i-1} B_i; B_i', \bar{A}_i B_1', \bar{A}_i^2 B_i', \cdots, \bar{A}_i^{r_i-1} B_i'$$

are independent numbers in $\bar{i}\bar{i}$ and may be used as $2r_i$ of the units. For, if

$$\sum_{p=0}^{r_i-1} c_p \bar{A}_i^p B_i + \sum_{p=0}^{r_i-1} c_p' \bar{A}_i^p B_i' = 0,$$

where the $c''$s are not all zero, we get on multiplying by the reciprocal of $\sum_{p=0}^{r_i-1} c_p' \bar{A}_i^p$, with respect to $I_i$, a linear relation between $B_i'$ and $B_i$, $\bar{A}_i B_i, \cdots, \bar{A}_i^{r_i-1} B_i$, which is contrary to hypothesis. If there is a number $B_i''$, in $\bar{i}\bar{i}$, independent of the $2r_i$ units now chosen, we proceed as before. By continuing this process we can show that the number $n_i$ of units in $\bar{i}\bar{i}$ is a multiple of $r_i$.

The numbers

$$\begin{array}{llllll} I_i, & \bar{A}_i, & \bar{A}_i^2, & \cdots, & \bar{A}_i^{r_i-1}; \\ A_i', & \bar{A}_i A_i', & \bar{A}_i^2 A_i', & \cdots, & \bar{A}_i^{r_i-1} A_i'; \\ A_i'^2, & \bar{A}_i A_i'^2, & \bar{A}_i^2 A_i'^2, & \cdots, & \bar{A}_i^{r_i-1} A_i'^2; \\ \cdot\ \cdot\ \cdot

which is contrary to hypothesis. If $B'_{ij}$ in $\overline{ij}$ is independent of

$$B_{ij},\ \bar{A}_i B_{ij},\ \bar{A}_i^2 B_{ij},\ \cdots,\ \bar{A}_i^{r_i-1} B_{ij},$$

we get the following $2r_i$ independent numbers in $\overline{ij}$.

$$B_{ij},\ \bar{A}_i B_{ij},\ \bar{A}_i^2 B_{ij},\ \cdots,\ \bar{A}_i^{r_i-1} B_{ij};\ B'_{ij},\ \bar{A}_i B'_{ij},\ \bar{A}_i^2 B'_{ij}\ \cdots,\ \bar{A}_i^{r_i-1} B'_{ij}.$$

Proceeding as in the proof of Theorem II, we may show that $n_{ij}$ is a multiple of $r_i$. Likewise it can be shown that $n_{ij}$ is a multiple of $r_j$ $(i = 0, 1, 2, \cdots, \bar{s};\ j = 1, 2, \cdots, \bar{s})$.

THEOREM V. *If the number of units in $\overline{i0}$ (or $\overline{0i}$) is just $r_i$ $(i = 1, 2, \cdots, \bar{s})$, then the product of any number in $\overline{i0}$ into any number in $\overline{00}$ (or any number in $\overline{00}$ into any number in $\overline{0i}$) is zero.*

Let $(i0)_1$ be any number in $\overline{i0}$. Then, as in the proof of Theorem IV, the numbers

$$(i0)_1,\ \bar{A}_i(i0)_1,\ \bar{A}_i^2(i0)_1,\ \cdots,\ \bar{A}_i^{r_i-1}(i0)_1$$

are independent and may be chosen as the $r_i$ units. Then for any number $(00)$ in $\overline{00}$,

$$(i0)_1(00) = [c_0 I_i + c_1 \bar{A}_i + c_2 \bar{A}_i^2 + \cdots + c_{r_i-1} A_i^{r_i-1}](i0)_1$$

since this product must also lie in $\overline{i0}$. From this follows

$$\left[\sum_{p=0}^{r_i-1} c_p \bar{A}_i^p\right]^m \cdot (i0)_1 = (i0)_1 \cdot (00)^m = 0.$$

Therefore, since $(i0)_1 \neq 0$, each $c_p = 0$. Thus

$$(i0)_1 \cdot (00) = 0,$$

whence follows the theorem.

THEOREM VI. *If there are units neither in $\overline{ij}$ nor in $\overline{ji}$, for $i = \alpha_1, \alpha_2, \cdots, \alpha_p$ and $j = \beta_1, \beta_2, \cdots, \beta_{p'}$, where $p + p' = \bar{s}$ and $\alpha_1, \alpha_2, \cdots, \alpha_p, \beta_1, \beta_2, \cdots, \beta_{p'}$ are all distinct, then the number system is reducible.*

For, under these conditions, all the units in the groups $\overline{ij}$ $(i, j = \alpha_1, \alpha_2, \cdots, \alpha_p)$, constitute a subsystem, and all the remaining units, viz., all those in the groups $\overline{ij}$ $(i, j = \beta_1, \beta_2, \cdots, \beta_{p'})$, constitute a second subsystem and these two subsystems are mutually nilfactorial.

THEOREM VII. *If $m = 1$ and*

$$n < \sum_{i=1}^{\bar{s}} r_i(m_i + 1) - r_j \qquad (j = 1, 2, \cdots, s),$$

*the system is reducible.*

Assuming that $m = 1$ and that the system is irreducible, we will find a lower limit to $n$. By Theorem I there is no unit in $\overline{00}$. There are units in at least one of the groups $\overline{\bar{s}\alpha}$ or $\overline{\alpha\bar{s}}$ for a positive integer $\alpha < \bar{s}$; for, if not, the system is reducible by Theorem VI, which is contrary to hypothesis. Let those groups

$\overline{\bar{s}\alpha_1}$ or $\overline{\alpha_1\bar{s}}$ $(\alpha_1 \neq \bar{s})$ that contain one or more units be given by

$$\alpha_1 = a_{11}, a_{12}, a_{13}, \cdots, a_{1p_1}.$$

If $p_1 < \bar{s} - 1$, then for some integer $a_2$ among $a_{11}, a_{12}, \cdots, a_{1p_1}$ there are units in $\overline{a_2\alpha_2}$ or $\overline{\alpha_2 a_2}$ for at least one integer $\alpha_2 (\alpha_2 < \bar{s})$ other than $a_{11}, a_{12}, \cdots, a_{1p_1}$. For if not, by Theorem VI the given system is reducible, which is contrary to hypothesis. Let the values of $\alpha_2$ giving those groups $\overline{a_2\alpha_2}$ or $\overline{\alpha_2 a_2}$ that contain one or more units be given by

$$\alpha_2 = a_{21}, a_{22}, \cdots, a_{2p_2},$$

where none of these integers are included among $\bar{s}, a_{11}, a_{12}, \cdots, a_{1p_1}$. We have $p_1 \geqq 1$, and we have shown that if $p_1 < \bar{s} - 1$, then $2 \leqq p_1 + p_2 \leqq \bar{s} - 1$. If $p_1 + p_2 < \bar{s} - 1$, then for some integer $a_3$ among $a_{11}, a_{12}, \cdots, a_{1p_1}$; $a_{21}, a_{22}, \cdots, a_{2p_2}$, there are units in $\overline{a_3\alpha_3}$ or $\overline{\alpha_3 a_3}$ for some integer $\alpha_3 < \bar{s}$ other than $a_{11}, a_{12}, \cdots, a_{1p_1}; a_{21}, a_{22}, \cdots, a_{2p_2}$. For otherwise, the system is reducible by Theorem VI; and $3 \leqq p_1 + p_2 + p_3 \leqq \bar{s} - 1$, where $\alpha_3 = a_{31}, a_{32}, \cdots a_{3p_3}$, none of these integers appearing among $a_{11}, a_{12}, \cdots, a_{1p_1}; a_{21}, a_{22}, \cdots, a_{2p_2}$. Repeating this process $t$ times for some integer $t \geqq 1$, we must have

$$p_1 + p_2 + p_3 + \cdots + p_t = \bar{s} - 1.$$

Thus each positive integer less than $\bar{s}$ and greater than 0 appears once and but once among the integers $a_{11}, a_{12}, a_{1p_1}; a_{21}, \cdots, a_{2p_2}; \cdots; \cdots; a_{t1}, a_{t2}, \cdots, a_{tp_t}$; provided 0 does not appear among these numbers (if 0 does appear $t$ must be chosen so that $p_1 + p_2 + \cdots + p_t = \bar{s}$ in which case all the integers $0, 1, 2, \cdots, \bar{s} - 1$ will appear once and but once); corresponding to each of these integers there is one group $\overline{\beta_i i}$, or $\overline{i\beta_i}$, or there are two groups $\overline{\beta_i i}$ and $\overline{i\beta_i}$ $(i = 1, 2, \cdots, \bar{s} - 1; \beta_i \neq i)$, which contain one or more units. Then, by Theorem III, the total number of units in these groups is not less than $\sum_{i=1}^{\bar{i}-1} r_i$.

By Theorem II, the total number of units in the groups $ii$ $(i = 1, 2, \cdots, \bar{s})$ is not less than $\sum_{i=1}^{\bar{i}} r_i m_i$, whence it follows that $n$ cannot be less than $\sum_{i=1}^{\bar{i}} r_i m_i + \sum_{i=1}^{\bar{i}-1} r_i$. That is, if

$$n < \sum_{i=1}^{\bar{i}} r_i (m_i + 1) - r_j \qquad (j = 1, 2, \cdots, \bar{s}),$$

the system is reducible.

THEOREM VIII. *If $m > 1$, and*

$$n < \sum_{i=1}^{\bar{i}} r_i (m_i + 1) + r_{\bar{i}} + m - 1,$$

*where the subscripts have been so chosen that $r_{\bar{i}} \leqq r_k$ $(k = 1, 2, \cdots, \bar{s})$, the system is reducible.*

Assuming the system to be irreducible, and repeating the process in the demonstration of Theorem VII, we find that units exist in a group $\overline{\beta_i i}$ or $\overline{i\beta_i}$,

or in two groups $\overline{\beta_i i}$ and $\overline{i\beta_i}$ for $i$ successively equal to $0, 1, 2, \cdots, \bar{s}-1$ $(\beta_i \neq i)$. Denote this set of groups by $\Gamma$. For $i \neq 0$ there are in the group $\overline{\beta_i i}$ (or $\overline{i\beta_i}$), at least as many as $r_i$ units; and for $i = 0$ there are at least as many as $r_{\beta_0}$ units in the group.

Now if there are more than $\bar{s}$ groups in $\Gamma$, there must be two groups for at least one value of $i$, as $i = \alpha$. And therefore in this case the total number of units in $\Gamma$ is not less than

$$r_{\beta_0} + \sum_{i=1}^{\bar{s}-1} r_i + r_\alpha \,(\alpha \neq 0), \qquad r_{\beta_0} + \sum_{i=1}^{\bar{s}-1} r_i + r_{\beta_0}\,(\alpha = 0).$$

If there are just $\bar{s}$ groups in $\Gamma$ and units in one or more groups $\overline{kl}$ $(k \neq l)$, not in $\Gamma$, then the total number of units in all the groups $\overline{ij}$ $(i \neq j)$ is not less than

$$r_{\beta_0} + \sum_{i=1}^{\bar{s}-1} r_i + r_k (k \neq 0), \qquad r_{\beta_0} + \sum_{i=1}^{\bar{s}-1} r_i + r_l\,(k = 0).$$

If there are just $\bar{s}$ groups in $\Gamma$ and units in no group $\overline{kl}$ $(k \neq l)$, not in $\Gamma$, the only existing group $\overline{0\rho}$ (or $\overline{\rho 0}$) is the single group $\overline{0\beta_0}$ (or $\overline{\beta_0 0}$) in $\Gamma$; and therefore if each number in it is nilfactorial to every number $\overline{00}$, the system is reducible. But this holds unless the group $\overline{0\beta_0}$ (or $\overline{\beta_0 0}$) contains $2r_{\beta_0}$ units. For, otherwise, by Theorem V, we have

$$(00)\cdot(0\beta_0) = 0\,[\text{or } (\beta_0 0)\cdot(00) = 0].$$

Therefore in this case the total number of units in $\Gamma$ is not less than

$$\sum_{i=1}^{\bar{s}-1} r_i + 2r_{\beta_0}.$$

Hence it follows, since $r_s \leqq r_k\,(k = 1, 2, \cdots, \bar{s})$, that in any case the total number of units in the groups $\overline{ij}$ $(i, j = 0, 1, 2, \cdots, \bar{s};\ i \neq j)$ is not less than

$$\sum_{i=1}^{\bar{s}} r_i + r_s.$$

By Theorem II the total number of units in the groups $\overline{ii}$ $(i = 1, 2, \cdots, \bar{s})$ is not less than $\sum_{i=1}^{\bar{s}} r_i m_i$. By Theorem III the number of units in the group $\overline{00}$ is not less than $m - 1$. Therefore, in any irreducible number system where $m > 1$, $r_s \leqq r_i$ $(i = 1, 2, \cdots, \bar{s})$,

$$n \leqq \sum_{i=1}^{\bar{s}} r_i (m_i + 1) + r_s + m - 1.$$

Hence the theorem is proved.

I shall next enumerate the number systems belonging to an arbitrary domain $R$ to which, or to the reciprocals of which, all irreducible number systems in less than five units, and belonging to $R$, may be transformed by linear transformations rational in $R$.

## § 3. *Nilpotent systems*, $n \leqq 4$.

If there is no idempotent number in a system it is nilpotent and all numbers in it are nilpotent.*

The multiplication tables of all the nilpotent number systems in $n$ units $e_1, e_2, \cdots, e_n$, to which, or to the reciprocals of which, all nilpotent systems in $n$-units can be transformed by transformations rational in $R$, can be found from those systems of the same kind in $n-1$ units $e_1, e_2, \cdots, e_{n-1}$, by adding to each product $e_i e_j = \sum_{k=1}^{n-1} \gamma_{ijk} e_k$ of units in the multiplication table of the system $e_1, e_2, \cdots, e_{n-1}$, a term consisting of $\gamma_{ij} e_n$, where $\gamma_{ij}$ is a parameter that remains to be determined; the product of any number of the system $e_1, e_2, \cdots, e_n$ and $e_n$ used either as a prefactor or a postfactor being written zero.†

Some of these parameters can be fixed by using the associative law for multiplication, and by simple transformations in which the new parameters are always found by solving linear equations (thus, no irrationality is introduced by these new parameters). For $n \leqq 4$, all the irreducible nilpotent number systems belonging to any domain can be shown equivalent to, or equivalent to a reciprocal of one or another of the systems whose multiplication tables are given below. When the domain $R$ is more precisely defined the number of these forms and the number of parameters in them may be reduced still further. All products not written are zero.

$n = 1.$ $e_1^2 = 0.$

$n = 2.$ $e_1^2 = e_2.$

$n = 3.$ $e_1^2 = e_3,\ e_2 e_1 = e_3,\ e_2^2 = a e_3;\ e_1^2 = e_3,\ e_2^2 = a e_3;\ e_1 e_2 = -e_2 e_1 = e_3;$
$e_1^2 = e_2,\ e_1 e_2 = e_2 e_1 = e_3.$

$n = 4.$ $e_1^2 = a e_4,\ e_2 e_3 = b e_4 = e_3^2,\ e_3 e_1 = e_4 = -e_3 e_2;\ a e_1^2 = e_2 e_1 = e_2^2 = e_3 e_2 = b e_3^2 = e_4;$
$e_1^2 = e_3 e_2 = e_4;\ a e_1^2 = e_2 e_1 = e_2^2 = b e_3^2 = e_4;\ a e_1^2 = e_2^2 = b e_3^2 = e_4;$
$e_1^2 = e_3,\ e_2 e_1 = e_4,\ e_1 e_2 = e_3 + a e_4,\ e_2^2 = b e_3 + c e_4;\ e_1^2 = e_3,\ e_2 e_1 = e_4,$
$e_2^2 = a e_3 + e_4;\ e_1^2 = e_3,\ a e_2^2 = e_2 e_1 = e_4;\ e_1^2 = e_3,\ e_2^2 = e_4;\ e_1^2 = e_3,$
$e_1 e_2 = a e_2 e_1 = e_2^2 = e_4;\ e_1^2 = e_3,\ e_1 e_2 = a e_2 e_1 = e_4;\ e_1^2 = a e_2^2 = e_3,$

* In accordance with what has been said above we refer here to nilpotent systems belonging to $R$, and to numbers lying in $\Re(R, e_i)$; for proof of this theorem see TABER, $T_1$, pp. 512, 525.

† Cf. H. E. HAWKES, Mathematische Annalen, vol. 58, p. 369.

In passing I note an error that appears in his paper. His Theorem VI, p. 369, states that "If two systems are deleted by the same method, ⋯ and the deleted systems are inequivalent, the original systems are inequivalent." But on deleting by his method the unit $e_4$ from each of the two equivalent systems,

$$e_1^2 = e_3,\ e_1 e_2 = e_4,\ e_2 e_1 = a e_4,\ e_2^2 = e_4 \quad \text{and} \quad e_1^2 = e_4,\ e_1 e_2 = e_3,\ e_2 e_1 = a e_3,\ e_2^2 = e_3,$$

we get the two inequivalent systems, $e_1^2 = e_3$ and $e_1 e_2 = e_3,\ e_2 e_1 = a e_3,\ e_2^2 = e_3$. (In these systems the products not written are zero.) The error in the proof given for this theorem lies in the failure to show that the determinant of the equations of transformation which, as HAWKES says, must exist between the units of the deleted systems, is not identically zero. That this can not be shown in general follows from the example just given.

$e_2e_1 = ae_1e_2 = e_4$; $e_1^2 = e_3$, $e_2^2 = ae_3 + e_4$, $e_1e_2 = ae_4$, $e_2e_1 = e_4$; $e_1^2 = e_3$, $e_1e_3 = ae_2e_1 = e_2^2 = e_3^2 = e_4$; $e_1^2 = e_3$, $e_1e_3 = e_4 = e_2e_1$; $e_i = e_1^i$ $(i = 1, 2, 3, 4)$.

§ 4. *Systems composed of a modulus and a nilpotent system.*

All systems in $n$ units or less, composed of a modulus and a nilpotent system, can be found either among the systems obtained by annexing a modulus to the nilpotent systems in less than $n$ units, or among the systems which are obtained by taking the units of two or more nilpotent systems, writing the products of any two units in different systems as zero and annexing a modulus to the nilpotent system thus formed. It is necessary to bring in these later systems since in the enumeration of nilpotent systems reducible systems were discarded and since the reducible nilpotent system produces, on having a modulus annexed, an irreducible system.

§ 5. *Enumeration of all number systems, in less than five units, belonging to an arbitrary domain R.*

Let $A$ be chosen as in § 1. If $\Omega(A) = 0$ is the fundamental equation of $A$, let us examine, for $n \leqq 4$, the possible forms of

$$\Omega(\lambda) \equiv \lambda^m [\Omega_1(\lambda)]^{m_1} [\Omega_2(\lambda)]^{m_2} \cdot \cdots \cdot [\Omega(\lambda)]^{m_{\bar{s}}},$$

$\Omega_i(\lambda)$ being of order $r_i$ in $\lambda$. By means of Theorems VII and VIII we find the possible combinations of values of $\bar{s}$, $m$, $m_i$, $r_i$ $(i = 1, 2, \cdots, \bar{s})$ for irreducible number systems in $n$ units; these are given in the following table:

| $n$ | $m$ | $\bar{s}$ | $r_1$ | $m_1$ | $r_2$ | $m_2$ |
|---|---|---|---|---|---|---|
| 1 | 2 | 0 | | | | |
| 1 | 1 | 1 | 1 | 1 | | |
| 2 | 3 | 0 | | | | |
| 2 | 2 | 0 | | | | |
| 2 | 1 | 1 | 2 | 1 | | |
| 2 | 1 | 1 | 1 | 2 | | |
| 2 | 1 | 1 | 1 | 1 | | |
| 3 | 4 | 0 | | | | |
| 3 | 3 | 0 | | | | |
| 3 | 2 | 0 | | | | |
| 3 | 1 | 2 | 1 | 1 | 1 | 1 |
| 3 | 1 | 1 | 3 | 1 | | |
| 3 | 1 | 1 | 1 | 3 | | |
| 3 | 1 | 1 | 1 | 2 | | |
| 3 | 1 | 1 | 1 | 1 | | |

| $n$ | $m$ | $\bar{s}$ | $r_1$ | $m_1$ | $r_2$ | $m_2$ |
|---|---|---|---|---|---|---|
| 4 | 5 | 0 | | | | |
| 4 | 4 | 0 | | | | |
| 4 | 3 | 0 | | | | |
| 4 | 2 | 1 | 1 | 1 | | |
| 4 | 2 | 0 | | | | |
| 4 | 1 | 2 | 1 | 2 | 1 | 1 |
| 4 | 1 | 2 | 1 | 1 | 1 | 1 |
| 4 | 1 | 1 | 4 | 1 | | |
| 4 | 1 | 1 | 2 | 2 | | |
| 4 | 1 | 1 | 2 | 1 | | |
| 4 | 1 | 1 | 1 | 4 | | |
| 4 | 1 | 1 | 1 | 3 | | |
| 4 | 1 | 1 | 1 | 2 | | |
| 4 | 1 | 1 | 1 | 1 | | |

To distinguish the various cases let $[abc_1^{d_1}c_2^{d_2}\cdots c_r^{d_r}; k]$ designate the $k$th number system among those for which $n=a$, $m=b$, $r_i=c_i$, $m_i=d_i$ $(i=1, 2, \cdots, \bar{s})$. All products omitted are zero.

The following systems are nilpotent and are found in § 3:

$[12; 1]$; $[23; 1]$, $[22; k]$; $[34; 1]$, $[33; k]$, $[32, k]$; $[45; 1]$, $[44; k]$, $[43; k]$, $[42; k]$.

The following systems made up of a modulus and a nilpotent system are found by means of § 4:

$[211^2; 1]$; $[311^3; 1]$, $[311^2; k]$ $(k>1)$, $[311^1; k]$ $(k>3)$; $[411^4; 1]$, $[411^3; k]$ $(k>1)$, $[411^2; k]$ $(k>5)$.

$[111^1; 1]$. $e_1^2 = e_1$.

$[212^1; 1]$. $e_1e_i = e_ie_1 = e_i$ $(1=1, 2)$, $e_2^2 = pe_1$ ($p$ = not-square in $R$).

When $R$ is the domain of real scalars we may put $p=-1$, getting the ordinary complex number system considered as belonging to the domain of real scalars.

$[211^1; 1]$. $e_1e_i = e_i$ $(i=1, 2)$.

$[311^11^1; 1]$. $e_1e_i = e_i$ $(i=1, 2)$, $e_ie_3 = e_i$ $(i=2, 3)$.

$[313^1; 1]$. $e_ie_1 = e_1e_i = e_i = e_2^{i-1}$ $(i=1, 2, 3)$, $e_2^3 - p_1e_2 - p_2I = 0$, where $\lambda^3 - p_1\lambda - p_2 = 0$ is irreducible in $R$.

$[311^2; 1]$. $e_1e_i = e_i$ $(i=1, 2, 3)$, $e_2e_1 = e_2$.

$[311^1; 1]$. $e_1e_i = e_i$ $(i=1, 2, 3)$.

$[311^1; 2]$. $e_1e_i = e_i$ $(i=1, 2)$, $e_3e_1 = e_3$.

$[421^1; 1]$. $e_1e_i = e_i$, $e_3e_i = e_{i+2}$ $(i=1, 2)$.

Here we have chosen $e_1 = I_1$, $e_2 = (10)$, $e_3 = (01)$, $e_4 = (00)$; by means of Theorems I and V, the products can easily be shown to be as written.

$[421^1; 2]$. $e_1e_i = e_i$ $(i=1, 2, 3)$ $e_2e_4 = e_3$.

Here we have chosen $e_1 = I_1$, $e_2 = (10)$, $e_3 = (10)\cdot(00)$, $e_4 = (00)$. If $e_3 = ae_2 = e_2e_4$, then $0 = e_3e_4 = ae_2e_4 = a^2e_2$; and therefore $a=0$, which makes the system reducible. That is, $e_2$ and $e_3$ may be properly chosen as independent units if the system is irreducible.

$[411^21^1; 1]$. $e_1e_i = e_i$ $(i=1, 2, 3)$, $e_ie_4 = e_i$ $(i=3, 4)$.

$[411^11^1; 1]$. $e_1e_i = e_i$, $e_2e_{i+2} = e_i$, $e_3e_i = e_{i+2}$, $e_4e_{i+2} = e_{i+2}$ $(i=1, 2)$.

$[411^11^1; 2]$. $e_1e_i = e_i$, $e_4e_{i+2} = e_{i+2}$ $(i=1, 2)$, $e_2e_4 = e_2$, $e_3e_1 = e_3$.

$[411^11^1; 3]$. $e_1e_i = e_i$ $(i=1, 2, 3)$, $e_ie_4 = e_i$ $(i=2, 3, 4)$.

$[414^1; 1]$. $e_ie_1 = e_1e_i = e_i = e_2^{i-1}$ $(i=1, 2, 3, 4)$, $e_2^4 - p_1e_2^2 - p_2e_2 - p_3I = 0$, $\lambda^4 - p_1\lambda^2 - p_2\lambda - p_3 = 0$ being irreducible in $R$.

$[412^2; 1]$. $e_ie_1 = e_1e_i = e_i$ $(i=1, 2, 3, 4)$, $e_2^2 = pe_1$, $\lambda^2 - p = 0$ being irreducible in $R$, $e_2e_3 = e_3e_2 = e_4$, $e_2e_4 = e_4e_2 = pe_2$.

In the systems $[412^1; k]$ we have

$$\Omega(\lambda) \equiv \lambda(\lambda^2 - p) \qquad (\lambda^2 - p \text{ irreducible in } R).$$

If one unit then two units outside $\overline{11}$ must exist and we have $[412^1; 1]$, $e_1 e_i = e_i$ $(i = 1, 2, 3, 4)$, $e_2 e_1 = e_2$, $e_2^2 = p e_1$, $e_2 e_3 = e_4$, $e_2 e_4 = p e_3$, or its reciprocal.

$[412^1; 2 \text{ and } 3.]$ These systems have all their units in the group $\overline{11}$. Since $n = 4$ there is a number $B$ in $\overline{11}$ linearly independent of $I_1$ and $A$. The fundamental equation of $A$ is

$$\Omega(A) = 0, \qquad [\Omega(\lambda) \equiv \lambda\Omega_1(\lambda) \equiv \lambda(\lambda^2 + p)].$$

Enlarge the domain $R$ by annexing to it $\lambda_1 = -\lambda_2$, the roots of $\lambda^2 + p = 0$, calling the enlarged domain $R'$. For the domain $R'$ the groups $G_{11}^{(1)}, G_{12}^{(1)}, G_{21}^{(1)}, G_{22}^{(1)}$ replace $\overline{11}$, and are characterized in $R'$ just as $\overline{11}, \overline{12}, \overline{21}, \overline{22}$ were in $R$; by transformation rational in $R'$ units can be so chosen that each will fall entirely into one or the other of these new groups. There are idempotent numbers $I_1^{(1)}$ and $I_2^{(1)}$ in $\mathfrak{R}(R'; e_i)$, and in $G_{11}^{(1)}$, $G_{22}^{(1)}$ respectively, such that in $\mathfrak{R}(R; e_i)$

$$I_1 = I_1^{(1)} + I_2^{(1)} \qquad \text{and} \qquad A = \lambda_1 I_1^{(1)} + \lambda_2 I_2^{(1)} = \lambda_1 (I_1^{(1)} - I_2^{(1)}).^*$$

If $I_1^{(1)} B I_1^{(1)} \neq 0$, then $I_2^{(1)} B I_2^{(1)} \neq 0$.

For if

$$I_1^{(1)} B I_1^{(1)} = \frac{-pB + ABA + \lambda_1(AB + BA)}{4p} \equiv B_1 \neq 0$$

while

$$I_2^{(1)} B I_2^{(1)} = \frac{-pB + ABA - \lambda_1(AB + BA)}{4p} = 0,$$

we have, on adding and subtracting,

$$\frac{-pB + ABA}{2p} = B_1 = \frac{\lambda_1(AB + BA)}{2p};$$

but since $A$ and $B$ lie in $\mathfrak{R}(R, e_i)$, while $\lambda_1$ does not lie in $R$, this equation can only hold if $AB + BA = 0$, and thus if $B_1 = 0$, which is contrary to hypothesis.

Let

$$\bar{B} \equiv B - (I_1^{(1)} B I_1^{(1)} + I_2^{(1)} B I_2^{(1)}).$$

The number $I_1^{(1)} B I_1^{(1)} + I_2^{(1)} B I_2^{(1)}$, which may be zero, lies in $\mathfrak{R}(R, e_i)$ and is commutative with $A$. For

$$I_1^{(1)} B I_1^{(1)} + I_2^{(1)} B I_2^{(1)} = \frac{-pB + ABA}{2p} = \frac{A(AB + BA)}{2p} = \frac{(AB + BA)A}{2p}$$

* TABER, $T_3$.

Then $\bar{B}$ lies in $\mathfrak{R}(R, e_i)$ and

$$I_1^{(1)}\bar{B}I_1^{(1)} = 0 = I_2^{(1)}\bar{B}I_2^{(1)}.$$

We have $\bar{B} \neq 0$. For if $\bar{B} = 0$ then $B$ is commutative with $A$; thus

$$B = \frac{-pB + ABA}{2p} = \frac{-pB + A^2B}{2p} = -B,$$

and $B$ is zero, which is contrary to hypothesis. Further $\bar{B}$, $I_1$, and $A$ are independent numbers in $\mathfrak{R}(R, e_i)$. For if

$$a\bar{B} + bI_1 + cA = 0 \qquad (a, b, c \text{ in } R),$$

we get on using $I_1^{(1)}$ as a prefactor and as a postfactor,

$$b + c\lambda_1 = 0,$$

that is

$$b = c = 0$$

since $b$ and $c$ are, while $\lambda_1$ is not, in $R$. Then $I_1$, $A$, $\bar{B}$, $A\bar{B}$, are linearly independent with respect to $R$, and may be used as units of our system.*

Let

$$B_{12} \equiv I_1^{(1)}\bar{B}I_2^{(1)}, \qquad B_{21} \equiv I_2^{(1)}\bar{B}I_1^{(1)}.$$

Then it can be shown, as in the case of $I_1^{(1)}BI_1'$ above, that if one of $B_{12}$ and $B_{21}$ is zero the other is also. But if $B_{12} = 0 = B_{21}$, then

$$\bar{B} = I_1\bar{B}I_1 = (I_1^{(1)} + I_1^{(2)})\bar{B}(I_1^{(1)} + I_2^{(1)}) = 0,$$

which is impossible. That is neither of $B_{12}$ and $B_{21}$ is zero. Then

$$\bar{B} = I_1\bar{B}I_1 = (I_1^{(1)} + I_2^{(1)})\bar{B}(I_1^{(1)} + I_2^{(1)}) = B_{12} + B_{21},$$

and therefore $A\bar{B} = -\bar{B}A$, since $A = \lambda_1(I_1^{(1)} - I_2^{(1)})$.

The number of units of our system considered as belonging to $R'$ certainly can not exceed the number of units of the system considered as belonging to $R$. But $I_1^{(1)}$, $B_{12}$, $B_{21}$, $I_2^{(1)}$ are linearly independent in $\mathfrak{R}(R'; e_i)$ and may be chosen as units in this domain. Then

$$B_{12}B_{21} = \alpha_{12}I_1^{(1)} \qquad (\alpha_{12} \text{ in } R'),$$

$$B_{21}B_{12} = \alpha_{21}I_2^{(1)} \qquad (\alpha_{21} \text{ in } R'),$$

using the scalar function as developed by H. TABER,† we have

$$\frac{\alpha_{12}}{2} = S(B_{12}B_{21}) = S(B_{21}B_{12}) = \frac{\alpha_{21}}{2}.$$

* See proof of Theorem II.

† TABER, $T_1$, p. 515.

Therefore

$$\bar{B}^2 = (B_{12} + B_{21})^2 = \alpha_{12}(I_1^{(1)} + I_2^{(1)}) = \alpha_{12} I_1$$

and $\alpha_{12}$ is in $R$ since $\bar{B}$ and $I_1$ are in $\Re(R, e_i)$. We have two cases $\alpha_{12} = 0$ and $\alpha_{12} \neq 0$.

$[412^1; 2]$. $e_1 e_i = e_i e_1 = e_i (i = 1, 2, 3, 4)$, $e_2^2 = -pe_1$, $e_2 e_3 = -e_3 e_2 = e_4$, $-e_2 e_4 = e_4 e_2 = pe_3$ ($\lambda^2 + p = 0$ irreducible in $R$).

$[412^1; 3]$. $e_1 e_i = e_i e_1 = e_i (i = 1, 2, 3, 4)$, $e_2^2 = -pe_1$, $e_3^2 = -qe_1$, $e_4^2 = -pqe_1$, $e_2 e_3 = -e_3 e_2 = e_4$, $-e_2 e_4 = e_4 e_2 = pe_3$, $e_3 e_4 = -e_4 e_3 = qe_2$ ($\lambda^2 + p = 0$ and $\lambda^2 + q = 0$ irreducible in $R$).

If $R$ is the domain of real scalars, $p$ and $q$ may be taken as unity; there results the system of Hamiltonian quaternions.

$[411^3; 1]$. $e_1 e_i = e_i e_1 = e_i (i = 1, 2, 3)$, $e_1 e_4 = e_4$, $e_2^2 = e_3$.

$[411^2; 1]$. $e_1 e_i = e_i (i = 1, 2, 3, 4)$, $e_2 e_1 = e_2$.

$[411^2; 2]$. $e_1 e_i = e_i (i = 1, 2, 3, 4)$, $e_2 e_1 = e_2$, $e_2 e_3 = e_4$.

$[411^2; 3]$. $e_1 e_i = e_i (i = 1, 2, 3)$, $e_i e_1 = e_i (i = 1, 2, 4)$, $e_3 e_4 = e_2$.

$[411^2; 4]$. $e_1 e_i = e_i (i = 1, 2, 3)$, $e_i e_1 = e_i (i = 1, 2, 4)$.

$[411^2; 5]$. $e_1 e_i = e_i e_1 = e_i (i = 1, 2, 3)$, $e_1 e_4 = e_4$.

$[411^1; 1]$. $e_1 e_i = e_i (i = 1, 2, 3, 4)$.

$[411^1; 2]$. $e_1 e_i = e_i (i = 1, 2, 3)$, $e_i e_1 = e_i (i = 1, 4)$.

Every irreducible hypercomplex number system in less than five units belonging to the domain of rationality $R$ may be transformed by a transformation rational in $R$ either to one of the above systems or to the reciprocal of one of them.

CLARK UNIVERSITY,
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# ON THE ASYMPTOTIC CHARACTER OF THE SOLUTIONS OF CERTAIN LINEAR DIFFERENTIAL EQUATIONS CONTAINING A PARAMETER*

BY

GEORGE D. BIRKHOFF

The aim of the present paper is to develop the asymptotic character of the solutions of linear differential equations of the form

$$\frac{d^n z}{dx^n} + \rho a_{n-1}(x, \rho)\frac{d^{n-1} z}{dx^{n-1}} + \cdots + \rho^n a_0(x, \rho) z = 0$$

for large $|\rho|$. The functions $a_i(x, \rho)$ are analytic in the complex parameter $\rho$ at $\rho = \infty$ and have derivatives of all orders in the real variable $x$. SCHLESINGER† has proved asymptotic properties for $\lim \rho = \infty$ on some fixed ray $\arg \rho = \alpha$; in this paper we prove similar properties for a region $\theta \leqq \arg \rho \leqq \psi$, but by a different method. In 1837 LIOUVILLE‡ treated the important special case

$$\frac{d^2 z}{dx^2} + [\rho^2 + g(x)] z = 0$$

when $\rho$ is real, the first problem of the kind to be considered; the method of attack used in this paper is of a similar nature.

It is purposed to make an application of the results here obtained to boundary value and expansion problems in a second paper.

I desire to make acknowledgment of the kind encouragement and valuable suggestions received from Professor E. H. MOORE, for both papers.

We consider functions $z(x, \rho)$ of a real variable $x$ on the closed interval

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† Mathematische Annalen, vol. 63 (1907), pp. 277–300. This paper appeared after the writing of the present paper. See also a paper by J. HORN, Mathematische Annalen, vol. 52 (1899), pp. 271–292.

‡ Liouville's Journal, vol. 2, p. 16. Cf. in particular § 3.

$(a, b)$ and of a complex parameter $\rho$, $|\rho| > R$, which satisfy a linear differential equation of the form

$$\frac{d^n z}{dx^n} + \rho a_{n-1}(x, \rho)\frac{d^{n-1}z}{dx^{n-1}} + \cdots + \rho^n a_0(x, \rho)z = 0. \tag{1}$$

Here we assume that the functions $a_i(x, \rho)$ are bounded by the inequalities

$$|a_i(x, \rho)| \leqq M \qquad (a \leqq x \leqq b,\ |\rho| > R), \tag{2}$$

and that

$$a_i(x, \rho) = \sum_{j=0}^{\infty} a_{ij}(x)\rho^{-j}. \qquad (|\rho| > R). \tag{3}$$

The coefficients $a_{ij}$ of this last series we assume to be continuous with continuous derivatives of all orders on $(a, b)$, but not necessarily real. As a consequence of (2) and (3) the $a_i(x, \rho)$ are continuous in $x$ and $\rho$, for the series (3) are series of continuous functions of $x$ and $\rho$ converging uniformly for $|\rho| \geqq R_0 > R$. Further we postulate that the $n$ roots

$$w_1(x),\ w_2(x),\ \cdots,\ w_n(x)$$

of the equation

$$w^n + a_{n-1,0}(x)w^{n-1} + \cdots + a_{00}(x) = 0 \tag{4}$$

are distinct for every $x$ on $(a, b)$.

By $R(\mu)$ we denote the real part of the complex number $\mu$.

DEFINITION. *By a region $S$ of the $\rho$-plane we understand a region for which the indices* 1 *to* $n$ *can be so arranged that*

$$R[\rho w_1(x)] \leqq R[\rho w_2(x)] \leqq \cdots \leqq R[\rho w_n(x)] \tag{5}$$

*for every $x$ on $(a, b)$ and every $\rho$ on $S$.*

If $\rho = \rho_0$ is an $S$ point, so that (5) obtains for $\rho = \rho_0$, it is clear that (5) holds if only $\arg \rho = \arg \rho_0$. By virtue of this fact the half line $\arg \rho = \arg \rho_0$ belongs to the same region $S$. *For a given $x$* the relations (5) define a certain *closed* sector,

$$\theta_x \leqq \arg \rho \leqq \psi_x,$$

containing the half line $\arg \rho = \arg \rho_0$. The largest closed sector common to all these sectors,

$$\theta \leqq \arg \rho \leqq \psi, \tag{6}$$

forms the region $S$, degenerating to a half line when $\theta = \psi$. The fundamental theorem refers to the solutions of (1) on these regions $S$, when such regions exist.

An important set of inequalities on a region $S$ is

$$\left|e^{\rho\int_a^\beta w_1(t)dt}\right| \leqq \left|e^{\rho\int_a^\beta w_2(t)dt}\right| \leqq \cdots \leqq \left|e^{\rho\int_a^\beta w_n(t)dt}\right| \quad (a \leqq a \leqq \beta \leqq b). \tag{7}$$

In an expression of the form

$$e^{\rho\theta(x)}[e_0(x)\rho^{-k} + e_1(x)\rho^{-k-1} + \cdots]$$

the term

$$e_0(x)\rho^{-k}$$

shall be called the *principal term.*

Finally we introduce the notation

$$\phi^{[k]}(x, \rho) = \rho^{-k}\frac{\partial^k}{\partial x^k}\phi(x, \rho),$$

so that (1) becomes

$$A(z) \equiv z^{[n]} + a_{n-1}(x, \rho)z^{[n-1]} + \cdots + a_0(x, \rho)z = 0; \tag{8}$$

and also the notation $E$ as a generic notation for functions of $\rho$ and other variables bounded for $|\rho| \geqq R^0$, when $R^0$ is sufficiently large.

LEMMA I. *For every value of $i$ from 1 to $n$ inclusive there exist an infinite number of functions $u_{i0}(x)$, $u_{i1}(x)$, $\cdots$, continuous and with continuous derivatives of all orders, such that $u_{i0}(x)$ does not vanish at any point of $(a, b)$ and such that if the functions*

$$u_i(x, \rho) = e^{\rho\int_a^x w_i(t)dt}\sum_{j=0}^{m-1} u_{ij}(x)\rho^{-j}$$

*be substituted for $z$ in the expression $A(z)$, the coefficient of*

$$e^{\rho\int_a^x w_i(t)dt}\rho^{-s} \qquad \begin{pmatrix} i=1, 2, \cdots, n \\ s=0, 1, \cdots, m \end{pmatrix}$$

*in the expression thus obtained vanishes identically.*

To prove this, write

$$u(x, \rho) = e^{\rho\int_a^x w(t)dt}v(x, \rho),$$

where $w(x)$ is some definite one of the $n$ roots $w_i(x)$. We find then

$$u^{[1]}(x, \rho) \equiv e^{\rho\int_a^x w(t)dt}[w(x)v(x, \rho) + v^{[1]}(x, \rho)],$$

$$u^{[2]}(x, \rho) \equiv e^{\rho\int_a^x w(t)dt}\left[\left([w(x)]^2 + \frac{1}{\rho}\frac{d}{dx}w(x)\right)v(x, \rho) + 2w(x)v^{[1]}(x, \rho) + v^{[2]}(x, \rho)\right],$$

and, in general,

$$u^{[j]}(x, \rho) \equiv e^{\rho\int_a^x w(t)dt}\sum_{k=0}^{j}\alpha_{jk}(x, \rho)v^{[k]}(x, \rho), \tag{9}$$

in which $\alpha_{jk}(x, \rho)$ is a polynomial in $1/\rho$ of degree $j-1$ at most, whose coefficients are functions of $x$ continuous on $(a, b)$ and with continuous derivatives of all orders. We note that the principal term in the coefficient of $v(x, \rho)$ in (9) is

$$\alpha_{j00}(x) = [w(x)]^j, \tag{10}$$

and the principal term in the coefficient of $v^{[1]}(x, \rho)$ is

$$\alpha_{j10}(x) = j[w(x)]^{j-1}; \tag{11}$$

also we note that

$$\alpha_{jj}(x, \rho) = 1.$$

Hence we conclude

$$A[u(x, \rho)] \equiv e^{\rho \int_a^x w(t)dt} \bar{A}[v(x, \rho)] \tag{12}$$

where

$$\bar{A}(z) \equiv z^{[n]} + \bar{a}_{n-1}(x, \rho) z^{[n-1]} + \cdots + \bar{a}_0(x, \rho) z, \tag{13}$$

the $\bar{a}_j(x, \rho)$ being conditioned as the $a_j(x, \rho)$ are in equations (2) and (3). For convenience we write

$$\bar{a}_n(x, \rho) = 1, \qquad \bar{a}_l(x, \rho) = 0 \qquad (l > n).$$

We see from (10), (11) that if we place

$$\bar{a}_i(x, \rho) \equiv \sum_{j=0}^{\infty} \bar{a}_{ij}(x) \rho^{-j},$$

then

$$\bar{a}_{00}(x) = [w(x)]^n + a_{n-1,0}(x)[w(x)]^{n-1} + \cdots + a_{00}(x) = 0 \tag{14}$$

and

$$\bar{a}_{10}(x) = n[w(x)]^{n-1} + (n-1)a_{n-1,0}(x)[w(x)]^{n-2} + \cdots + a_{10}(x), \tag{15}$$

so that $\bar{a}_{10}(x) \neq 0$ on $(a, b)$, the $n$ roots $w_i(x)$ being distinct.

If then we write in (12)

$$v(x, \rho) = \sum_{j=0}^{m-1} u_j(x) \rho^{-j},$$

we find the condition that the coefficient of

$$e^{\rho \int_a^x w(t)dt} \rho^{-s} \qquad (s = 0, 1, \cdots, m)$$

in $A[u(x, \rho)]$ vanishes to be

$$\sum_{j+k+l=s} \bar{a}_{ij}(x) \frac{d^l}{dx^l} u_k(x) = 0. \tag{16}$$

The equation (16) is true for $s = 0$ by (14). For $s \neq 0$ we can write (16) in the form

$$\bar{a}_{10}(x) \frac{d}{dx} u_{s-1}(x) + \bar{a}_{01}(x) u_{s-1}(x) + \sum_{j+k+l=s}^{k<s-1} \bar{a}_{ij}(x) \frac{d^l}{dx^l} u_k(x) = 0, \tag{17}$$

since, if in (16) $k = s$, then $l = j = 0$ and the term corresponding to this set of values has a coefficient $\bar{a}_{00}(x) = 0$; if $k = s - 1$, we have either $l = 1$, $j = 0$ or $l = 0$, $j = 1$, and the corresponding terms are the first terms of (17).

It appears then that $u_{s-1}(x)$ can be determined in terms of $u_{s-2}(x), u_{s-3}(x), \cdots, u_0(x)$ as a solution of a certain linear differential equation of the first order which has no singular points on $(a, b)$ since $\bar{a}_{10}(x) \neq 0$. Thus $u_0(x), u_1(x), \cdots, u_{m-1}(x)$ are obtained in succession from (16) for $s = 1, 2, \cdots, m$. For each $w_i(x)$ we obtain in this way a sequence of functions $u_{i0}(x), u_{i1}(x), \cdots$, such that if the expressions

$$u_i(x, \rho) = e^{\rho \int_a^x w_i(t)dt} \sum_{j=0}^{m-1} u_{ij}(x)\rho^{-j}$$

be substituted for $z$ in $A(z)$, the coefficient of

$$e^{\rho \int_a^x w_i(t)dt} \rho^{-s} \qquad (i = 1, 2, \cdots, n\,;\ s = 0, 1, \cdots, m)$$

vanishes by (17) since the conditions (16) are now satisfied for $s = 1, 2, \cdots, m$. Furthermore the differential equation for $u_{i0}(x)$ is homogeneous, so that by taking for $u_{i0}(x)$ a solution which is different from zero at one point of $(a, b)$, we are sure that $u_{i0}(x)$ does not vanish at any point of $(a, b)$. Since the $a_{ij}(x)$ were continuous with all of their derivatives, the functions $u_{i0}(x), u_{i1}(x), \cdots$ are also continuous with all of their derivatives. The sequence of functions $u_{i0}(x), u_{i1}(x), \cdots$ has then the properties stated in the lemma.

If the formal developments

$$e^{\rho \int_a^x w_i(t)dt} \sum_{j=0}^{\infty} u_{ij}(x)\rho^{-j} \qquad (i = 1, 2, \cdots, n)$$

converged and admitted of $n$-fold term by term differentiation, we should have in them the asymptotic developments of solutions of our differential equation which we desire. This however will not in general be the case. We shall, nevertheless, show in the theorem below that by breaking off after $m$ terms of these series (i. e., by retaining the part $u_i(x, \rho)$) and putting in certain remainder terms we can get true solutions of our differential equation. In order to prove this we must first prove another lemma in which the form of the differential equation is established which is satisfied by the sums of these $m$ terms (i. e., $u_i(x, \rho)$) without any remainder:

LEMMA II. *The homogeneous linear differential equation of order $n$ with $n$ solutions $u_i(x, \rho)$ has the form*

$$B(z) \equiv z^{[n]} + b_{n-1}(x, \rho)z^{[n-1]} + \cdots + b_0(x, \rho)z = 0,$$

*where for $|\rho| > R^0$*

$$(18) \qquad |b_i(x, \rho)| \leqq M^0; \qquad b_i(x, \rho) = \sum_{j=0}^{\infty} b_{ij}(x)\rho^{-j} \qquad (i = 0, 1, \cdots, n-1).$$

*The coefficients $b_{ij}(x)$ which appear here are continuous together with all their derivatives, and*

$$b_{ij}(x) = a_{ij}(x) \quad (i=0, 1, \cdots, n-1; j=0, 1, \cdots, m). \tag{19}$$

The homogeneous linear differential equation of order $n$ with solutions $u_i(x, \rho)$ is

$$\begin{vmatrix} z^{[n]} & z^{[n-1]} & \cdots & z \\ u_1^{[n]}(x, \rho) & u_1^{[n-1]}(x, \rho) & \cdots & u_1(x, \rho) \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ u_n^{[n]}(x, \rho) & u_n^{[n-1]}(x, \rho) & \cdots & u_n(x, \rho) \end{vmatrix} = 0. \tag{20}$$

For the elements of this determinant we have

$$u_i^{[j]}(x, \rho) = e^{\rho \int_a^x w_i(t)dt} \sum_{k=0}^{m+j-1} \lambda_{ijk}(x)\rho^{-k} \tag{21}$$

where by (10)

$$\lambda_{ij0}(x) = [w_i(x)]^j u_{i0}(x). \tag{22}$$

Thus if we factor out of (20)

$$\prod_{i=1}^{n} e^{\rho \int_a^x w_i(t)dt}$$

the differential equation takes the form

$$\beta_n(x, \rho)z^{[n]} + \beta_{n-1}(x, \rho)z^{[n-1]} + \cdots + \beta_0(x, \rho)z = 0, \tag{23}$$

where $\beta_i(x, \rho)$ are polynomials in $1/\rho$. We have for the principal term of $\beta_n(x, \rho)$,

$$\beta_{n0}(x) = \begin{vmatrix} \lambda_{1,n-1,0} & \lambda_{1,n-2,0} & \cdots & \lambda_{100} \\ \lambda_{2,n-1,0} & \lambda_{2,n-2,0} & \cdots & \lambda_{200} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \lambda_{n,n-1,0} & \lambda_{n,n-2,0} & \cdots & \lambda_{n00} \end{vmatrix}.$$

In view of (22) this last determinant may be written

$$\prod_{i=1}^{n} u_{i0}(x) \cdot \begin{vmatrix} [w_1(x)]^{n-1} & [w_1(x)]^{n-2} & \cdots & w_1(x) & 1 \\ [w_2(x)]^{n-1} & [w_2(x)]^{n-2} & \cdots & w_2(x) & 1 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ [w_n(x)]^{n-1} & [w_n(x)]^{n-2} & \cdots & w_n(x) & 1 \end{vmatrix}.$$

This is not zero at any point on $(a, b)$ as the $w_i(x)$ are distinct and the $u_{i0}(x) \neq 0$ by Lemma I. Therefore for $|\rho| \geqq R^0$

$$\left|\frac{\beta_i(x,\rho)}{\beta_n(x,\rho)}\right| \leqq M^0; \qquad b_i(x,\rho) = \frac{\beta_i(x,\rho)}{\beta_n(x,\rho)} = \sum_{j=0}^{\infty} b_{ij}(x)\rho^{-j}.$$

But we can write (23) in the form

$$z^{[n]} + b_{n-1}(x,\rho)z^{[n-1]} + \cdots + b_0(x,\rho)z = 0. \tag{24}$$

Now the functions $b_{ij}(x)$ and their derivatives are continuous since the $w_i(x)$ and $u_i(x)$ are of this character. The first part of the lemma is thus proved.

Now let $j_0$ be the smallest value of $j$ for which, for some $i$ and $x$,

$$b_{ij}(x) \neq a_{ij}(x).$$

From (21) and (22) we see that the principal term of

$$B[u_i(x,\rho)] - A[u_i(x,\rho)] \equiv \overline{B-A}[u_i(x,\rho)] \tag{25}$$

is

$$\rho^{-j_0}\left[\sum_{k=0}^{n-1}[b_{kj_0}(x) - a_{kj_0}(x)][w_i(x)]^k\right]u_{i0}(x). \tag{26}$$

Assume $j_0 \leqq m$ if possible. In each part of the difference (25) the coefficient of

$$e^{\rho\int_a^x w_i(t)dt}\rho^{-j_0}$$

must then vanish, in the first part since $u_i(x,\rho)$ are solutions of $B(z) = 0$; in the second part, by Lemma I. Therefore from (26)

$$\sum_{k=0}^{n-1}[b_{kj_0}(x) - a_{kj_0}(x)][w_i(x)]^k = 0 \qquad (i = 1, 2, \cdots, n).$$

We conclude that

$$b_{kj_0}(x) = a_{kj_0}(x) \qquad (k = 0, 1, \cdots, n-1),$$

the $w_i(x)$ being distinct. This is a contradiction. Hence $j_0 > m$.

THEOREM. *On a region $S$ there exist $n$ independent solutions*

$$z_1(x,\rho),\ z_2(x,\rho),\ \cdots,\ z_n(x,\rho)$$

*of*

$$\frac{d^n z}{dx^n} + \rho\, a_{n-1}(x,\rho)\frac{d^{n-1}z}{dx^{n-1}} + \cdots + \rho^n a_0(x,\rho)z = 0$$

*analytic in $\rho$ such that if the integer $m$ is chosen at pleasure and $\rho$ is on $S$,*

$$z_i(x,\rho) = u_i(x,\rho) + e^{\rho\int_a^x w_i(t)dt}\cdot E_0\rho^{-m},$$

$$\frac{d}{dx}z_i(x,\rho)=\frac{d}{dx}u_i(x,\rho)+e^{\rho\int_a^x w_i(t)dt}\cdot E_1\rho^{-m+1}, \tag{27}$$

. . . . . .

$$\frac{d^{n-1}}{dx^{n-1}}z_i(x,\rho)=\frac{d^{n-1}}{dx^{n-1}}u_i(x,\rho)+e^{\rho\int_a^x w_i(t)dt}\cdot E_{n-1}\rho^{-m+n-1},$$

*where*

$$u_i(x,\rho)=e^{\rho\int_a^x w_i(t)dt}\sum_{j=0}^{m-1}u_{ij}(x)\rho^{-j}$$

*and* $u_{i0}(x)$ *does not vanish at any point of* $(a,b)$.

*Proof.* We proceed to effect a comparison of the solutions of the given differential equation with the solutions of the differential equation of Lemma II. First we write the given differential equation in the form

$$\frac{d^nz}{dx^n}+\rho b_{n-1}(x,\rho)\frac{d^{n-1}z}{dx^{n-1}}+\cdots+\rho^n b_0(x,\rho)z=\rho^n\overline{B-A}(z). \tag{28}$$

The development of the coefficient of $z^{[k]}$ in $\overline{B-A}(z)$ begins with a term in $\rho^{-m-1}$, in view of (18) and (19), so that

$$|b_k(x,\rho)-a_k(x,\rho)|\leqq D\cdot|\rho|^{-m-1} \tag{29}$$

for $|\rho|\geqq R^0$.

The general solution of a non-homogeneous linear differential equation

$$\frac{d^ny}{dx^n}+k_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+k_0(x)y=\phi(x) \tag{30}$$

may be written* in terms of a set of linearly independent solutions $y_1(x), y_2(x), \cdots, y_n(x)$ of the reduced equation in the form

$$y=\sum_{i=1}^n c_iy_i(x)+\int_a^x\left[\sum_{i=1}^n y_i(x)z_i(\xi)\right]\phi(\xi)d\xi, \tag{31}$$

where the functions $z_1(x), z_2(x), \cdots, z_n(x)$ are determined from the conditions

$$\sum_{i=1}^n\left(\frac{d^l}{dx^l}y_i(x)\right)z_i(x)=\begin{cases}0 & (l=0,1,\cdots,n-2),\\ 1 & (l=n-1),\end{cases}$$

while $c_1, c_2, \cdots, c_n$ are arbitrary constants. Any $y$ and $\phi$ satisfying (30) also satisfy (31) for some choice of $c_1, c_2, \cdots, c_n$; and if $y$ and $\phi$ satisfy (31) for some $c_1, c_2, \cdots, c_n$, they also satisfy (30); hence (30) and (31) are entirely equivalent.

* Cf., for instance, SCHLESINGER: *Handbuch der Theorie der linearen Differentialgleichungen*, vol. 1, p. 78.

Thus treating (28) as a non-homogeneous equation we find an *equivalent equation*

$$(32)\quad z(x,\rho)=\sum_{i=1}^{n}c_iu_i(x,\rho)+\int_a^x\left[\sum_{i=1}^{n}u_i(x,\rho)v_i(\xi,\rho)\right]\rho^n\overline{B-A}[z(\xi,\rho)]\,d\xi$$

where the $v_i$'s are obtained from the equations

$$\sum_{i=1}^{n}u_i^{[l]}(x,\rho)v_i(x,\rho)=\begin{cases}0 & (l=0,1,\cdots,n-2),\\ \rho^{-n+1} & (l=n-1).\end{cases}$$

If in place of these functions $v_i$ we introduce functions $\bar{u}_i$ defined by the equation

$$v_i(x,\rho)=\rho^{-n+1}\bar{u}_i(x,\rho),$$

equation (32) becomes

$$(33)\quad z(x,\rho)=\sum_{i=1}^{n}c_iu_i(x,\rho)+\rho\int_a^x\left[\sum_{i=1}^{n}u_i(x,\rho)\bar{u}_i(\xi,\rho)\right]\overline{B-A}[z(\xi,\rho)]\,d\xi,$$

where the $\bar{u}_i$ are determined from the equations

$$(34)\qquad \sum_{i=1}^{n}u_i^{[l]}(x,\rho)\bar{u}_i(x,\rho)=\begin{cases}0 & (l=0,1,\cdots,n-2),\\ 1 & (l=n-1).\end{cases}$$

In order to prove the existence of a solution $z_k(x,\rho)$ of the character stated in the theorem for some definite $k$ we make the final transformation of the constants $c_i$ in (33),

$$(35)\qquad \begin{aligned} c_i&=c_i' && (i=1,2,\cdots,k),\\ c_i&=-\rho\int_a^b\bar{u}_i(\xi,\rho)\,\overline{B-A}[z(\xi,\rho)]\,d\xi+c_i' && (i=k+1,k+2,\cdots,n).\end{aligned}$$

This transformation is reversible. The given differential equation thus appears finally in the form of an *equivalent* integral equation

$$(36)\qquad \begin{aligned} z(x,\rho)=\sum_{i=1}^{n}c_i'u_i(x,\rho)&+\rho\int_a^x\left[\sum_{i=1}^{k}u_i(x,\rho)\bar{u}_i(\xi,\rho)\right]\overline{B-A}[z(\xi,\rho)]\,d\xi\\ &+\rho\int_b^x\left[\sum_{i=k+1}^{n}u_i(x,\rho)\bar{u}_i(\xi,\rho)\right]\overline{B-A}[z(\xi,\rho)]\,d\xi,\end{aligned}$$

that is, the solutions of (36) and $A(z)=0$ are the same. In the form (33) we could infer one and but one solution $z(x,\rho)$ for a given set of $c_1,c_2,\cdots,c_n$; in fact, that solution of $A(z)=0$ which satisfies the conditions

$$\frac{d^j}{dx^j}z(a,\rho)=\sum_{i=1}^{n}c_i\frac{d^j}{dx^j}u_i(a,\rho)\qquad(j=0,1,\cdots,n-1),$$

but a similar inference is not possible for (36).

Write

$$\sum_{i=1}^{k} u_i^{[l]}(x,\rho)\bar{u}_i(\xi,\rho) = e^{\rho\int_\xi^x w_k(t)dt}\phi_{kl}(x,\xi;\rho),$$

$$\sum_{i=k+1}^{n} u_i^{[l]}(x,\rho)\bar{u}_i(\xi,\rho) = e^{\rho\int_\xi^x w_k(t)dt}\psi_{kl}(x,\xi;\rho). \tag{37}$$

For the consideration of (36) it is fundamental to establish that the functions $\phi_{kl}$ and $\psi_{kl}$ are bounded in the following manner for $|\rho| \geqq R^0$:

$$\begin{aligned} &|\phi_{kl}(x,\xi;\rho)| \leqq T, \qquad a \leqq \xi \leqq x \leqq b \\ &|\psi_{kl}(x,\xi;\rho)| \leqq T, \qquad a \leqq x \leqq \xi \leqq b \end{aligned} \qquad (l=0,1,\cdots,n-1). \tag{38}$$

To prove this we note first that from (21)

$$u_i^{[l]}(x,\rho) = e^{\rho\int_a^x w_i(t)dt}\eta_{il}(x,\rho), \tag{39}$$

in which $\eta_{il}(x,\rho)$ denotes a polynomial in $1/\rho$. If now we substitute in (34)

$$\bar{u}_i(x,\rho) = e^{-\rho\int_a^x w_i(t)dt}\bar{\eta}_i(x,\rho), \tag{40}$$

we find for the determination of $\bar{\eta}_i(x,\rho)$ the $n$ linear equations

$$\sum_{i=1}^{n}\eta_{il}(x,\rho)\bar{\eta}_i(x,\rho) = \begin{cases} 0 & (l=0,1,\cdots,n-2), \\ 1 & (l=n-1). \end{cases} \tag{41}$$

The determinant of these equations is a polynomial in $1/\rho$ which we called $\beta_n(x,\rho)$, and has the principal term $\beta_{n0}(x)$ [see (23)] which is not zero. We conclude that for $|\rho| \geqq R^0$

$$|\eta_{il}(x,\rho)| \leqq \eta, \qquad |\bar{\eta}_i(x,\rho)| \leqq \eta. \tag{42}$$

Now from (39) and (40) we have

$$|u_i^{[l]}(x,\rho)\bar{u}_i(\xi,\rho)| \leqq |e^{\rho\int_\xi^x w_i(t)dt}\eta^2|, \tag{43}$$

whence

$$|u_i^{[l]}(x,\rho)\bar{u}_i(\xi,\rho)| \leqq |e^{\rho\int_\xi^x w_k(t)dt}\eta^2|, \tag{44}$$

if we make the restriction

$$\xi \leqq x \quad (i=1,2,\cdots,k), \qquad \xi \geqq x \quad (i=k+1,k+2,\cdots,n).$$

To see this one recalls the inequalities (7). Hence

$$\begin{aligned} &\left|\sum_{i=1}^{k} u_i^{[l]}(x,\rho)\bar{u}_i(\xi,\rho)\right| \leqq k\left|e^{\rho\int_\xi^x w_k(t)dt}\eta^2\right| && (\xi \leqq x), \\ &\left|\sum_{i=k+1}^{n} u_i^{[l]}(x,\rho)\bar{u}_i(\xi,\rho)\right| \leqq (n-k)\left|e^{\rho\int_\xi^x w_k(t)dt}\eta^2\right| && (\xi \geqq x), \end{aligned} \tag{45}$$

which are in effect inequalities (38).

We now consider that solution $z_k(x, \rho)$ of (36) for which

$$c_i' = \begin{cases} 0 & (i \neq k), \\ 1 & (i = k), \end{cases} \tag{46}$$

and we will show

($a$) that for $|\rho| \geqq R'$ one and but one such solution exists, and this is analytic in $\rho$;

($b$) that the solutions $z_1, z_2, \cdots, z_n$ thus defined fulfill the relations (27) of our fundamental theorem.

The linear independence of $z_1, z_2, \cdots, z_n$ is then an immediate consequence of their form (27), since $u_1, u_2, \cdots, u_n$ are linearly independent. The demonstration is thus completed.

*Proof of* ($a$).

We know there exists one and but one solution of (33) for all sets of values of $c_i$. To each set corresponds a definite transformed set $c_i'$ which we will show has the form

$$c_i' = \sum_{j=1}^{n} \gamma_{ij}(\rho) c_j,$$

where the $\gamma_{ij}(\rho)$ are analytic in $\rho$. To prove this statement we define $Z_k(x, \rho)$, $(k = 1, 2, \cdots, n)$, to be that solution of $A(z) = 0$ which satisfies the equations

$$\frac{d^j}{dx^j} Z_k(a, \rho) = \frac{d^j}{dx^j} u_k(a, \rho) \qquad (j = 0, 1, \cdots, n-1).$$

If $z$ is the solution of (32) for the set $c_1, c_2, \cdots, c_n$, we have then

$$z = \sum_{j=1}^{n} c_j Z_j(x, \rho).$$

The $Z_k(x, \rho)$ are analytic in $\rho$ since the coefficients in $A(z) = 0$ and the $u_i(x, \rho)$ are. If this value of $z$ is substituted in (35), we obtain the transformation in the stated form.

From this we see that either a unique solution for the set $c_i'$ of (46) exists, in which case this solution will be analytic in $\rho$, or there is a solution of (36) for $c_i' = 0$ $(i = 1, 2, \cdots, n)$. If then we prove that the latter alternative is impossible for $|\rho| > R'$, we shall have proved statement ($a$).

Let us now write down (36) and the equations obtained from it by differentiation, using the definitions (37). We then obtain

$$\begin{aligned} z^{[l]}(x, \rho) = \sum_{i=1}^{n} c_i' \cdot u_i^{[l]}(x, \rho) &+ \rho \int_a^x e^{\rho \int_\xi^x w_k(t)dt} \phi_{kl}(x, \xi; \rho) \overline{B - A}\,[z(\xi, \rho)] d\xi \\ &+ \rho \int_b^x e^{\rho \int_\xi^x w_k(t)dt} \psi_{kl}(x, \xi; \rho) \overline{B - A}\,[z(\xi, \rho)]\, d\xi \end{aligned} \tag{47}$$

$$(l = 0, 1, \cdots, n-1).$$

These are a set of linear integral equations in $z(x, \rho)$, $z^{[1]}(x, \rho)$, $\cdots$, $z^{[n-1]}(x, \rho)$ of the FREDHOLM type.

Assume if possible a solution $z(x, \rho)$ to exist for $c_i' = 0$ $(i = 1, 2, \cdots, n)$. If we write

$$z^{[l]}(x, \rho) = e^{\rho \int_a^x w_k(t)dt} z_l(x, \rho) \qquad (l = 0, 1, \cdots, n-1),$$

the equations (47) become

$$(48) \quad z_l(x, \rho) = \rho \int_a^x \phi_{kl}(x, \xi; \rho) g[z(\xi, \rho)] d\xi + \rho \int_b^x \psi_{kl}(x, \xi; \rho) g[z(\xi, \rho)] d\xi \qquad (l = 0, 1, \cdots, n-1)$$

where

$$(49) \qquad g[z(x, \rho)] \equiv \sum_{j=0}^{n-1} [b_j(x, \rho) - a_j(x, \rho)] z_j(x, \rho).$$

If $W \neq 0$ be the maximum of

$$|z_l(x, \rho)| \qquad (l = 0, 1, \cdots, n-1)$$

on $(a, b)$, we conclude from (29) that for $|\rho| \geqq R^0$

$$(50) \qquad |g[z(\xi, \rho)]| \leqq nD \cdot W \cdot |\rho|^{-m-1}.$$

But in one of the equations (48), e. g., $l = l_1$, $|z_l(x, \rho)|$ has the value $W$ at $x = x_1$, so that

$$(51) \quad W = \left| \rho \int_a^{x_1} \phi_{kl_1}(x_1, \xi; \rho) g[z(\xi, \rho)] d\xi + \rho \int_b^{x_1} \psi_{kl_1}(x_1, \xi; \rho) g[z(\xi, \rho)] d\xi \right|.$$

Applying to (51) inequalities (50) and (38), as is possible since $\xi \leqq x_1$ in $\phi_{kl_1}(x_1, \xi; \rho)$ of (38) $\xi \geqq x_1$ and in $\psi_{kl_1}(x_1, \xi; \rho)$, we find

$$(52) \qquad W \leqq n(b - a) T \cdot D \cdot W \cdot |\rho|^{-m}$$

which is not possible for

$$|\rho| \geqq \sqrt[m]{n(b - a) T \cdot D}.$$

Thus the set of values $c_i' = 0$ is seen to be impossible for $|\rho| \geqq R'$. Hence a unique solution for (46) exists.

*Proof of (b).*

By (46) we have

$$(53) \quad z_k^{[l]}(x, \rho) = u_k^{[l]}(x, \rho) + \rho \int_a^x \left[ \sum_{i=1}^{k} u_i^{[l]}(x, \rho) \bar{u}_i(\xi, \rho) \right] \overline{B - A}[z_k(\xi, \rho)] d\xi + \rho \int_b^x \left[ \sum_{i=k+1}^{n} u_i^{[l]}(x, \rho) \bar{u}_i(\xi, \rho) \right] \overline{B - A}[z_k(\xi, \rho)] d\xi.$$

Writing then

$$z_k^{[l]}(x, \rho) = e^{\rho \int_a^x w_k(t)dt} z_{kl}(x, \rho),$$

we obtain

$$z_{kl}(x,\rho) = \eta_{kl}(x,\rho) + \rho\int_a^x \phi_{kl}(x,\xi;\rho)g[z_k(\xi,\rho)]\,d\xi + \rho\int_b^x \psi_{kl}(x,\xi;\rho)g[z_k(\xi,\rho)]\,d\xi, \tag{54}$$

where $\eta_{kl}(x,\rho)$ and $g(z)$ are defined as in (39) and (49). This equation (54) is analogous to (48).

Let $W$ denote the maximum of

$$|z_{kl}(x,\rho)| \qquad (l=0,1,\cdots,n-1)$$

on $(a,b)$. Then we find in analogy with (50) that for $|\rho| \geqq R^0$

$$|g[z_k(x,\rho)]| \leqq nD\cdot W\cdot|\rho|^{-m-1}. \tag{55}$$

If this maximum be attained for $l=l_1$, $x=x_1$, we find in analogy with (52)

$$W \leqq |\eta_{kl_1}(x_1,\rho)| + n(b-a)\,T\cdot D\cdot W|\rho|^{-m},$$

whence, if $m>0$ and $|\rho|$ is large,

$$W \leqq Q, \tag{56}$$

since

$$|\eta_{kl_1}(x_1,\rho)| \leqq \eta.$$

But from (54)

$$|z_{kl}(x,\rho)-\eta_{kl}(x,\rho)| = \left|\rho\int_a^x \phi_{kl}(x,\xi;\rho)g[z_k(\xi,\rho)]\,d\xi + \rho\int_b^x \psi_{kl}(x,\xi;\rho)g[z_k(\xi,\rho)]\,d\xi\right|.$$

Therefore, using (38) and the inequality

$$|g[z_k(x,\rho)]| \leqq nD\cdot Q|\rho|^{-m-1},$$

which is a consequence of (56), we see that for a large enough $|\rho|$

$$|z_{kl}(x,\rho)-\eta_{kl}(x,\rho)| \leqq n(b-a)\,T\cdot D\cdot Q|\rho|^{-m}.$$

Recalling that $E$ is a generic notation for functions of $\rho$ and other variables bounded for large $|\rho|$, we conclude from this at once that

$$z_k(x,\rho) = u_k(x,\rho) + e^{\rho\int_a^x w_k(t)dt}E_0\cdot\rho^{-m},$$

$$\frac{d}{dx}z_k(x,\rho) = \frac{d}{dx}u_k(x,\rho) + e^{\rho\int_a^x w_k(t)dt}E_1\cdot\rho^{-m+1},$$

. . . . . .

$$\frac{d^{n-1}}{dx^{n-1}}z_k(x,\rho) = \frac{d^{n-1}}{dx^{n-1}}u_k(x,\rho) + \rho^{\rho\int_a^x w_k(t)dt}E_{n-1}\cdot\rho^{-m+n-1}.$$

# ON THE HOLOMORPH OF THE CYCLIC GROUP OF ORDER $p^m$ *

BY

G. A. MILLER

1. *Introduction.* The present article is a continuation of one published in this journal, vol. 4 (1903), p. 151, under the more general title, "On the holomorph of a cyclic group." It was observed that the holomorph of the cyclic group of any order is the direct product of the holomorphs of its Sylow subgroups, and hence the investigation of the holomorph of the general cyclic group may be divided into the consideration of the holomorph of the cyclic group of order $p^m$ and the study of the direct product of given groups. The present article is devoted to some of the important questions under the first topic, which have not been explicitly treated in the earlier paper. The holomorph of a cyclic group not only plays a fundamental rôle in many group theoretic discussions, but also presents questions equivalent to those arising in the theory of exponents to which numbers belong with respect to a given arbitrary modulus $m$. For instance, the numbers which have recently been called † the primitive roots of $m$ are those which correspond to the operators of highest order in the group of isomorphisms of the cyclic group of order $m$ and hence the determination of the number of such primitive roots is a special case of the determination of the number of operators of a given order in an abelian group. ‡

2. *Cyclic groups of order* $2^m$, $m > 3$. We begin with the case when $p = 2$ and shall assume that the cyclic group $(H)$ of order $2^m$, $m > 3$, is represented as a regular substitution group. The holomorph $(G)$ of $H$ is composed of all the substitutions in these $2^m$ letters which transform $H$ into itself. Let $s_1$ represent a substitution § of order $2^{m-2}$ which generates all those substitutions of the group of isomorphisms $(I)$ of $H$ which are commutative with the operators of order 4 in $H$. From the manner in which $s_1$ transforms the substitutions of $H$, it is evident that $s_1$ is composed of 2 cycles of each of the orders

$$2^{m-2},\ 2^{m-3},\ \cdots,\ 4,\ 2.$$

Hence $s_1^{2^\alpha}$, $\alpha < m - 2$, contains $2^{\alpha+1}$ cycles of each of the orders

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† EPSTEIN, Archiv der Mathematik und Physik, vol. 12 (1907), p. 135.

‡ Annals of Mathematics (series 2), vol. 2 (1901), p. 77; vol. 6 (1904), p. 1.

§ Bulletin of the American Mathematical Society, vol. 7 (1901), p. 350.

$2^{m-\alpha-2}, 2^{m-\alpha-3}, \cdots, 2$. The number of the substitutions of a given order in the group $\{H, s_1\}$, generated by $H$ and $s_1$, is exactly equal to this order whenever the degree is less than $2^m$. That is, $\{H, s_1\}$ contains two such substitutions of order 2, four of order 4, etc., and all those of the same order are conjugate under $\{H, s_1\}$. Hence the total number of the substitutions of $\{H, s_1\}$ whose degree is less than $2^m$ is

$$1 + 2 + 4 + \cdots + 2^{m-2}$$

and all its remaining substitutions are regular. The totality of the substitutions of $\{H, s_1\}$ whose orders divide $2^\alpha (\alpha = 1, 2, \cdots, m)$ constitutes a characteristic subgroup under $G$.

To obtain $G$ we may extend $\{H, s_1\}$ by means of a substitution $(s_2)$ which transforms into their inverse the operators of order 4 contained in $H$ and involves all the letters found in $s_1$. Then $\{s_1, s_2\}$ is the group of isomorphisms of $H$ and it is of degree $2^m - 2$. All its substitutions except those generated by $s_1$ are of this degree and each of them is conjugate with $2^{m-1}$ substitutions under $G$. The characteristic subgroup of $G$ which is generated by its substitutions which are squares is of index 8 under $G$. Hence $G$ contains exactly seven subgroups of half its own order. In other words, *the holomorph of the cyclic group of order* $2^m$, $m > 3$, *contains exactly seven subgroups of half its own order.*

3. *The auxiliary groups* $I$ *and* $I_0$. The holomorph contains only one invariant operator besides the identity and hence its group of cogredient isomorphisms $(I_0)$ is of order $2^{2m-2}$. We proceed to determine some of the properties of $I_0$ with a view to a complete determination of the group of isomorphisms $(I)$ of $G$, the group of isomorphisms of $H$ being well known. As $I_0$ has a $(1, 2)$ isomorphism with $G$ it contains two invariant cyclic subgroups corresponding to $H$ and its conjugate under $I$. The operator of $I_0$ which corresponds to $s_1$ in this isomorphism is of the same order as $s_1$. The subgroup of $I_0$ which corresponds to $\{H, s_1\}$ is therefore similiar to $\{H, s_1'\}$, where $s_1'$ transforms the operators of $H$ in exactly the same manner as $s_1$ does but the order of $s_1'$ is twice that of $s_1$. Hence $I_0$ involves three operators of order 2 which are invariant under $I$. One of these is generated by $s_1'$. The construction of $I_0$ is therefore like that of the holomorph of the cyclic group of order $2^{m-1}$ with the single exception that the operator which corresponds to $s_1$ is of order $2^{m-2}$ instead of being of order $2^{m-3}$. In other words, we may construct $I_0$ by constructing the holomorph of the cyclic group of order $2^{m-1}$ and then replacing the operator which corresponds to $s_1$ by the product of this operator and an operator of order $2^{m-2}$ which is independent of and hence commutative with every operator of this holomorph. Hence it results that *the group of cogredient isomorphisms of the holomorph of the cyclic group of order* $2^m$ *is a* $(2, 2^m)$ *isomorphism between the cyclic group of order*

*$2^{m-2}$ and the holomorph of the cyclic group of order $2^{m-1}$ with respect to its dihedral subgroup involving this cyclic subgroup.*

It is known that the order of $I$ is four times that of $I_0$ and hence it remains to find operators which transform the operators of $G$ among themselves and are not in $I_0$. One such operator $(s_3)$ of order 2 is commutative with $s_2$ and with all the operators of $\{H, s_1\}$ which are commutative with the operators of order 8 in $H$ while it transforms $s_1$ into itself multiplied by the operator of order 2 in $H$. As the operators of $G$ which are commutative with $s_3$ form a characteristic subgroup, $s_3$ is invariant under $I$ and hence $I$ contains at least 7 invariant operators of order 2 and *the total number of operators which transform both $H$ and $G$ into themselves constitute the direct product of $I_0$ and an operator of order* 2.

4. *The group $I$ continued.* To complete the determination of $I$ it is only necessary to find a substitution $(s_4)$ which transforms $G$ into itself and a generator of $H$ into itself multiplied by $s_1^{m-3}$. It is clear that $s_4$ transforms the square of such a generator into itself multiplied by the operator of order 2 in $H$ and that we may assume that $s_4^{-1} s_2 s_4 = s_2 s_1^{m-3}$. As an operator of $I$, $s_4$ transforms the operators of $I_0$ which are of highest order into themselves multiplied by the invariant operator under $I$ which corresponds to $s_1^{m-3}$ in $I_0$. It also transforms the operator which corresponds to $s_2$ in $I_0$ into itself multiplied by the same operator of $I_0$. Hence $s_4$ is of order 2 and has two conjugates under $I$. Moreover, $I$ is the direct product of $\{I_0, s_4\}$ and the group of order 2 generated by $s_3$ since $s_3$ and $s_4$ are commutative. Hence *$I$ contains* 8 *and only* 8 *invariant operators and these constitute the group of type* $(1, 1, 1)$.

It is easy to determine the orders of all the operators of $I$ by means of the given properties of $I_0$ and of $s_3$ and $s_4$. Just one-fourth of the operators of $I_0$ are of order $2^{m-1}$ and these correspond to the operators of order $2^m$ in $\{H, s_1\}$. The 3 invariant operators of order 2 in $I_0$ correspond to operators of $\{H, s_1\}$ and the $2^m$ non-invariant operators of this order correspond to other operators of $G$. Hence the number of operators of order 2 in $I_0$ is $2^m + 3$. The number of the operators of order $2^\alpha (1 < \alpha < m-1)$ is given by the expression $2^{m+\alpha-2} + 3.2^{2(\alpha-1)}$. From the properties of $s_4$ mentioned above it follows that the number of operators of each order in $\{I_0, s_4\}$ is the same as the number of such operators in the direct product of $I_0$ and an operator of order 2. This follows from the facts that $s_4$ transforms $2^{m-1}$ operators of order 4 in $I_0$ into their inverses and is also non-commutative with the same number of operators of order 2 in $I_0$. Hence it results that the number of the operators of order $2^\alpha (1 < \alpha < m)$ contained in $I$ is four times the number of the operators of the same order in $I_0$. From this it follows that the number of operators of order 2 in $I$ is 3 increased by four times the number of these operators in $I_0$.

5. *Cyclic group of order $2^m$, $m < 4$.* In the above considerations it was

assumed that $m \geqq 4$. When $m < 4$ the order of $G$ cannot exceed 32 and hence $G$ is well known and its $I$ has been determined. Hence we do not need to consider the special cases when $m < 4$.

6. *Alternative method of discussion.* Some of the above results can readily be deduced from the fact that $G$ contains two invariant dihedral groups of order $2^{m+1}$ which have $2^m$ common operators. Their non-invariant operators of order 2 may be divided into three distinct sets each of which contains $2^{m-1}$ operators. One of these sets is composed of such operators as are common to both of the invariant dihedral groups while each of the other sets occurs in only one of these groups. As $G$ contains only two operators which are commutative with each operator of the first of these sets as well as with each operator of one of the other sets it follows that $I_0$ may be represented as an intransitive substitution group of degree $2^m$ which is obtained by establishing a $(2, 2)$ isomorphism between the holomorph of the cyclic group of order $2^{m-1}$ written in two distinct sets of letters. From this it follows directly that $I_0$ contains exactly four invariant operators.

With respect to the three sets of $2^{m-1}$ operators of order 2 mentioned in the preceding paragraph, $I_0$ may be represented as an intransitive group of degree $3.2^{m-1}$ involving as its transitive constituents the holomorph of the cyclic group of order $2^{m-1}$ written in three distinct sets of letters. Two of these holomorphs are simply isomorphic and the group formed by this simply isomorphism has a $(2, 2)$ isomorphism with the third transitive constituent. The group $\{I_0, s_4\}$ may therefore be represented as a $(4, 2)$ isomorphism between an imprimitive group of degree $2^m$ and the holomorph of the cyclic group of order $2^{m-1}$ while $I$ is the direct product of this intransitive group and the group of order 2. This intransitive group is simply isomorphic with the group of cogredient isomorphisms of the double holomorph of $H$ and hence *the group of isomorphisms of the holomorph of the cyclic group of order* $2^m$ *is the direct product of the group of order* 2 *and the group of cogredient isomorphisms of the double holomorph of the cyclic group of order* $2^m$.

7. *Cyclic groups of order* $p^m$, $p > 2$. When $p > 2$ the group of isomorphisms of $H$ is cyclic and $G$ is a complete group. Hence the considerations for this more general value of $p$ are very much simpler than those for the special value $p = 2$. In view of its greater simplicity, this case seems to require no further developments than those found in the paper mentioned at the beginning of this article, with the exception of a few results which are of especial interest in contrast with results for $p = 2$. When $H$ is represented as a regular group, its group of isomorphisms, which is known to be cyclic and of order $p^{m-1}(p-1)$, contains a subgroup of order $p^{m-1}$ composed of all its substitutions which are commutative with an operator of order $p$ in $H$. A generator of this subgroup is composed of $p-1$ cycles of each of the orders $p^{m-1}, p^{m-2}, \cdots, p$; and each of

the substitutions of this group of order $p^{m-1}(p-1)$ which is not in this subgroup is of degree $p^m - 1$. Hence the group of isomorphisms of $H$ is of degree $p^m - 1$ when $p > 2$ and of degree $p^m - 2$ when $p = 2$.

Moreover, when $p = 2$ the group of isomorphisms of $H$ as a subgroup of degree $2^m - 2$ corresponds to a subgroup of degree $2^m$ in some holomorphism of $G$; while it must always correspond to a subgroup of its own degree when $p > 2$. In every case there are exactly $p^m$ subgroups with which a particular subgroup transforming $H$ into all its holomorphisms may correspond in some holomorphism of $G$. From the fact that, when $p > 2$, the maximal subgroup which omits a given letter is of degree $p^m - 1$ and cannot correspond to a subgroup of a different degree in a holomorphism of $G$, it follows that the group of isomorphisms of $G$ can be represented as a transitive group involving $G$ and hence we have a very simple proof of the known fact that $G$ is a complete group.*

*Messenger of Mathematics, vol. 37 (1907), p. 55.

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# ON NON-MEASURABLE SETS OF POINTS, WITH AN EXAMPLE*

BY

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LEBESGUE'S theory of integration is based on the notion of the *measure* of a set of points, a notion introduced by BOREL and subsequently refined by LEBESGUE himself. The question whether there are non-measurable sets of points was taken up by LEBESGUE † a few months ago in a memoir bearing the title *Contribution a l'étude des correspondances de M. Zermelo.* While yet unaware that the question of their existence was here considered, I constructed recently in quite a different manner a very simple example of a non-measurable set, an example which has some especial interest when considered after LEBESGUE in the light of correspondences. To give this example is the object of § 1. It is obtained by splitting up the unit interval into a countable set of points and two superimposable sets of points of upper measure 1. Each of these in turn can be split into two non-measurable sets (§ 2), also of upper measure 1, and so on.

§ 2 gives a brief, general analysis of non-measurable sets of points, which so far as I know is entirely new. It is shown that when any two complementary non-measurable linear point-sets are resolved each into two components, one of which is a subset of maximum measure § and the other of which is not measurable, the sum of the two non-measurable components is measurable and its measure is equal to the upper measure of either component.

Finally, the intimate connection between the upper and lower measures of any linear set is revealed by showing that not only is the lower measure equal to the measure of a subset of maximum measure, but *the difference between the upper and lower measures is equal to the upper measure of the residual non-measurable component* (Theorem 2).

## § 1. *Example of a non-measurable set of points.*

Let $S$ denote any set of points within the interval (0, 1), and let $C$ be the complementary set. The upper measure of $S$ — call it $\overline{m}(S)$ — is the lower

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† LEBESGUE, Bulletin de la societé mathematique de France, vol. 35 (1907), p. 212.

§ Not necessarily unique.

limit to the sum of a countable set of intervals inclosing $S$ in their interior. If in the set a number of intervals successively overlap, they can be combined into a single one. Hence a set of non-overlapping intervals may be used, the extremities of which are points of $C$. To inclose a point set we shall use henceforth only sets of intervals whose terminal points belong to the complementary point set.

Suppose now that $\overline{m}(S) < 1$. Then $S$ can be inclosed in a set of non-overlapping intervals whose sum $1 - \Sigma$ is as nearly equal to $\overline{m}(S)$ as we please, and the points exterior to the interior of these intervals form a closed subset of $C$ having a measure equal to $\Sigma$.

I shall now restrict $S$ and $C$ by the following condition: In $(a, b)$, any subinterval whatsoever of $(0, 1)$, there can be found an interval as nearly equal in length to $ab$ as we please and such that the subsets of $C$ and $S$ included in it shall be *similar* to $C$ and $S$. In other words, $C$ and $S$ can be obtained from their subsets in this portion of $ab$ by mere magnification. In place of a single portion of $ab$ having the property postulated it will serve my purpose as well if there is a finite number of such portions, the sum of whose lengths differs from $ab$ by as little as we please. This condition secures for $C$ and $S$ what I shall term a *homogeneous character*. A familiar instance will be obtained by selecting for $C$ the rational points in $(0, 1)$ or the points having the abscissas $m/2^n$, in which $m$ and $n$ are arbitrary positive integers.

*Under this restriction it can now be affirmed that either $S$ and $C$ are measurable and one of them has the measure* 0, *or they are both non-measurable and have an upper measure equal to* 1.

To prove this, suppose first that $\overline{m}(S) < 1$. Inclose $S$ in a first set of intervals of measure $1 - \Sigma < 1$. Then the points exterior to their interior form a first subset $C^{(1)}$ of $C$ having a measure $\Sigma$. Now in each interval of this first set of intervals we can find by hypothesis a subinterval (or a finite number of subintervals with a total length) as nearly equal to it in length as we please and such that the subset of $C$ which it contains is similar to $C$. Consequently in this subinterval (or in each of the finite-number of subintervals) there can be selected a set of intervals which is similar to the "first set" and which has its end points in $C$. The points of the subinterval (or subintervals) exterior to the interior of this set of intervals form a subset $C'$ of $C$ which is similar to $C^{(1)}$ and exterior to $C^{(1)}$. If every interval of the first set of intervals is treated in this way we obtain a second set of intervals entirely within the first, and the measure of this second set may be taken greater than $(1 - \epsilon)(1 - \Sigma)^2$, where $\epsilon$ denotes a small arbitrarily prescribed positive quantity. The subsets $C'$ in the first set of intervals which are excluded from the second set form, when taken together, a second subset $C^{(2)}$ of $C$ which has a measure greater than $(1 - \epsilon)(1 - \Sigma)\Sigma$. Let the second set of intervals be treated in the same manner as the first, and

so on. Continuing in this manner we find that $C$ contains distinct subsets, the measures of which are greater than the successive terms of the progression

$$\Sigma + (1-\epsilon)(1-\Sigma)\Sigma + (1-\epsilon)^2(1-\Sigma)^2\Sigma + \cdots = \frac{\Sigma}{\Sigma + \epsilon(1-\Sigma)}.$$

As $\epsilon$ is arbitrary, it follows that $C$ has a component with a measure as nearly equal to 1 as we please. Taking for $\epsilon$ a sequence of values approaching 0 as their limit, we obtain measurable components of $C$ whose sum has a measure equal to 1. The set of points in $(0, 1)$ complementary to this sum has the measure 0 and consists of the remaining points of $C$ and all of the points of $S$. Therefore we conclude that if $\overline{m}(S) < 1$, the sets $C$ and $S$ are measurable, and the latter has the measure 0.

If, on the other hand, $\overline{m}(C) < 1$, we may adjoin to $S$ the two terminal points 0, 1 and repeat the reasoning with the interchange of $S$ and $C$. In this repetition certain points in the $i$ successive sets of intervals $(i = 1, 2, 3, \cdots)$, which correspond to the terminal points 0, 1 of the initial set, must either be neglected or be adjoined to $S$. This, however, is absolutely without influence upon the reasoning, as these points are countable and the measure of a countable set of points is 0.

Lastly, if $\overline{m}(S) = \overline{m}(C) = 1$, neither set can be measurable since the sum of their upper measures exceeds the length of the unit-interval. We thus reach the conclusion announced that either $S$ and $C$ are measurable and the measure of one of them is 0, or they are both non-measurable with an upper measure equal to the length of the interval.

We are now ready to begin the construction of two non-measurable sets $S$ and $C$. For this purpose divide the unit-interval arbitrarily into pairs of points, $x$ and $y = 1 - x$, which are the images of each other with respect to the middle point of the interval. The middle point can be neglected or added to $S$ or $C$. Any two point sets $\{x\}$ and $\{y\}$ thus constructed must have equal upper measures, as either set, regarded as a rigid system, can be superimposed by reflection upon the other. If they are measurable, their common upper measure is $\frac{1}{2}$. To prevent this I shall secure for $S$ and $C$ the "homogeneous character" previously discussed.

To this end we will associate with each $x$ in the set $S$ the countable set $\{x \pm m/2^n\}$ and correspondingly with each $y$ in $C$ the set of points $\{1 - (x \pm m/2^n)\}$, in which $m$ and $n$ denote arbitrary positive integers. These two sets of points are entirely distinct unless $x$ itself and hence all points of both sets have the form $m'/2^{n'}$. Points with abscissas of this special form may either be assigned arbitrarily to $S$ and $C$ or neglected. Instead of so doing, however, we shall put aside these points into a separate point set $K$. This, being countable, has the measure 0 and its removal will in no wise affect $\overline{m}(S)$ or $\overline{m}(C)$.

The restriction just made upon the assignment of the points of the unit interval to $S$ and $C$ secures for these two point sets the following property. If the interval is divided into $2^n$ equal parts each of these parts, in respect to the distribution of the points of $S$ and of $C$, is an exact copy of every other part. Consequently, to secure their similarity to the initial unit interval, I have merely to require, if possible, that whenever a point $x$ is assigned to $S$ the points $x/2, x/4, x/8, \cdots$ shall be assigned to the same set, and conversely, $x$ shall be assigned simultaneously with any $x/2^n$. This requirement will give rise to a conflict in the distribution of the points of $(0, 1)$ to $S$ and $C$ if, and only if, a point $x/2^{p'} \pm m'/2^{n'}$ should coincide with one having the coördinate $1 - (x/2^{p''} \pm m''/2^{n''})$, in which $p', m', n', p'', m'', n''$ denote non-negative integers. But clearly this can occur only when $x$ is rational. Let all the rational points of the unit interval be thrown into our negligible set $K$ of measure 0. After the rational points are thus set aside, the points of the interval may be separated into pairs of sets of countable points,

$$(1) \qquad \left\{\frac{x}{2^{\pm p}} \pm \frac{m}{2^n}\right\}, \qquad \left\{1 - \left(\frac{x}{2^{\pm p}} \pm \frac{m}{2^n}\right)\right\} \quad (m, n, p = 0, 1, 2, \cdots),$$

to be assigned respectively to $S$ and $C$.

The homogeneous structure desired has now been obtained. Furthermore, when the unit interval is divided into any $2^n$ equal parts, these parts are copies of one another in respect to the distribution of points of $K$ as well as $S$ and $C$, and are similar to the whole interval. Consider now the two complementary sets $S$ and $C + K$. If any subinterval $(a, b)$ of $(0, 1)$ is taken, two points $a' = m_1/2^n$ and $b' = m_2/2^n$ may be found in the interior so near to $a$ and $b$ respectively that $ab - a'b'$ is as small as we please, and $a'b'$ consists of $m_2 - m_1$ equal portions of length $1/2^n$ which are similar to the initial unit interval in respect to the distribution of the points of $S$ and $C + K$. Since these two sets are of equal upper measure, it follows from the preceding discussion that *they are non-measurable and have an upper measure* 1. By subtraction of $K$ from $C + K$ the same is seen to hold for $C$.

In dividing thus the unit interval into two non-measurable sets the separation has been left indeterminate, in so far as it was permitted to assign arbitrarily one of two image points $x$ and $1 - x$ to $S$, each of the two points determining then one of the groups (1). To remove the arbitrary element of choice it is not at all necessary to give a law which will pick out uniquely a *principal element* from each group $(x/2^{\pm p} \pm m/2^n)$. This appears, in consequence of a remark of LEBESGUE,* to be impossible. Rather, the removal of the arbitrary feature depends on our ability to give a law which, out of an uncountable infinity of pairs of groups, will select one of each pair. Suppose now that we express any

*See LEBESGUE's footnote, loc. cit., p. 210.

point $x$ in the binary system; thus

$$x = .\alpha_1\alpha_2\alpha_3\cdots \qquad (\alpha_i = 0 \text{ or } 1).$$

Then

$$1 - x = .\beta_1\beta_2\beta_3\cdots \qquad (\alpha_i + \beta_i = 1).$$

Any other point $x/2^{\pm p} \pm m/2^n$ in the same group as $x$ will agree with $x$ in the succession of digits 0 and 1 after the omission of a finite number of initial elements, $\alpha_1', \alpha_2', \cdots, \alpha_k'$ and $\alpha_1, \cdots, \alpha_j$, in the two numbers compared. The problem, therefore, which presents itself is to make a choice between $x = .\alpha_1\alpha_2\cdots$ and $1 - x = .\beta_1\beta_2\cdots$ based *on the ultimate form* of the binary decimal and not at all on the initial elements. Such a choice can be made, if not for the continuum $(0, 1)$, at least for many subsets. For example, suppose the ratio of the number of zeros to the number of 1's in $.\alpha_1\alpha_2\cdots\alpha_n$ to have as the abscissas of its largest and smallest points of condensation, for increasing $n$, the values $A$ and $B$. Then the corresponding values for $1 - x$ are $1/B$ and $1/A$. Unless $A = 1/B$ it is clearly possible to give a rule for selecting one point from each pair, $x$ and $1 - x$. Take now in the unit-interval the set which consists of all points for which $A \neq 1/B$. This has a homogeneous character and is its own image with respect to $x = \frac{1}{2}$. If its upper measure is not 0, it may replace the unit-interval in our previous reasoning, and can be split up into two non-measurable superimposable subsets in precisely the same way as the unit interval. Thus it seems to me possible, and perhaps not difficult, to remove the arbitrary element of choice in my example by confining one's attention to a proper subset of the continuum, though as yet I have not succeeded in proving that this is possible.

It remains to ascertain to what categories the sets $S$ and $K$ belong. By BAIRE's definition a point set of the *first category* is one that can be created by putting together a countable number of sets, each of which is nowhere dense. Now $S$ and $C$ are superimposable, and this is true also for their subsets included in any portion $ab$ of our unit interval. This shows that they are throughout of the second category,* for otherwise some subinterval of the unit interval would be obtained as the sum of point sets, $S$, $C$, and $K$ of the first category, but this is impossible.

## § 2. *On the theory of non-measurable sets.*

In the preceding section an interval has been split up into two complementary non-measurable sets, *each of which has an upper measure equal to the measure of the sum.* That a similar decomposition always takes place whenever a measurable linear set is resolved into two non-measurable sets will appear from the following analysis.

* I use here the term second category to denote a set which is not of the first category, rather than in the sense of BAIRE. Cf. LEBESGUE, l. c., p. 212.

Take for consideration any two complementary non-measurable sets $S$ and $C$ in the unit interval. If $\overline{m}(S)$ and $\overline{m}(C)$ are not both equal to 1, suppose $\overline{m}(S) < 1$, and adjoin to $C$, if necessary, the points $x = 0$, $x = 1$. Then $S$ may be so inclosed within a set of intervals having their end points in $C$ that the closed sets of points exterior to the interior of these intervals shall have a measure differing from $1 - \overline{m}(S)$ by less than a small, arbitrarily assigned, positive $\epsilon$. Taking for $\epsilon$ a sequence of values having 0 as its limit we obtain a countable set of measurable closed subsets of $C$. The sum of these subsets has a measure not less than $1 - \overline{m}(S)$. Clearly also its measure can not exceed this amount, for otherwise the measure of the complementary set would be less than $\overline{m}(S)$, and hence also the upper measure of $S$ which is included in it. This shows that $C$ contains a subset $C_1$ of maximum measure $1 - \overline{m}(S)$. Similarly $S$ has a subset $S_1$ of maximum measure $1 - \overline{m}(C)$.

Place now

$$S = S_1 + S', \qquad C = C_1 + C'.$$

Together $S'$ and $C'$ make a measurable set $T$ such that

$$m(S_1) + m(C_1) + m(T) = 1. \tag{2}$$

But

$$1 = \overline{m}(S) + m(C_1), \qquad 1 = \overline{m}(C) + m(S_1), \tag{3}$$

and

$$\overline{m}(S) \leqq m(S_1) + \overline{m}(S'), \qquad \overline{m}(C) \leqq m(C_1) + \overline{m}(C'). \tag{4}$$

Multiplying equation (2) by 2 and then adding to them (3) and (4), we find that

$$2m(T) \leqq \overline{m}(S') + \overline{m}(C'). \tag{5}$$

Since $S'$ and $C'$ are subsets of $T$, this can be only if

$$\overline{m}(S') = \overline{m}(C') = m(T).$$

The analysis and result hold also when we assume $\overline{m}(S) = 1$, also if we split up $S$ and $C$ in any other way, each into a subset of maximum measure (necessarily the same maximum) and a non-measurable subset. We reach therefore the following result:

THEOREM 1. *If $S$ is any non-measurable linear aggregate and $C$ is its complement, let them each be resolved in any way into a subset of maximum measure and a non-measurable subset. Then the two non-measurable components taken together make a measurable set, and the upper measure of either component is equal to the measure of their sum.*

It is to be noticed that the use necessarily of the sign of equality in (5) carries with it the use of the same sign in (4). Or in other words:

THEOREM 2. *For any non-measurable set $S$ we have*

$$\overline{m}(S) = m(S_1) + \overline{m}(S'),$$

*in which $S_1$ denotes any subset of $S$ of maximum measure and $S'$ is the residual non-measurable subset.*

LEBESGUE's lower measure $\underline{m}(S)$ is by definition $1 - \overline{m}(C)$ or, by virtue of the last theorem, $1 - m(C_1) - \overline{m}(C')$. Since also $\overline{m}(C') = m(T)$, we see from equation (2) that *the lower measure $\underline{m}(S)$ of any linear point set $S$ is equal to the measure of the subset $m(S_1)$ of maximum measure which it contains.* This is equivalent to W. H. YOUNG's *definition of the inner content** of a set as "the upper limit of the content of its closed components."

Theorems 1 and 2 seem to me to advance considerably the theory of measure and of non-measurable point sets. In particular, theorem 2 throws new light on the connection between the upper and lower measures. YOUNG's definitions of the two measures, unlike LEBESGUE's, were of very diverse nature. Equation (2) shows that he, nevertheless, was concerning himself with the essential nature of a point-set. By treating the residue $S'$, neglected by YOUNG, I have brought out the close connection between the definitions of YOUNG, and the development has been simple and rapid, connecting closely with both LEBESGUE and YOUNG.

Several corollaries† can be developed rapidly from theorems 1 and 2. For example, it follows at once that *the upper measure of the sum of a measurable and non-measurable set which have no point in common is the sum of their upper measures.*

I scarcely need to remark that any subset of the non-measurable component $S'$ is likewise non-measurable, unless, of course, it is of measure 0.

The inference should not be hastily made on the basis of theorem 1 that a new and more comprehensive definition of measure is possible; to wit, as the measure of the maximum measurable component plus one-half of the irreducible residue. For such a definition it is not true, in general, that the measure of the sum of two sets without a common point is the sum of their measures. To see this it will suffice to show that just as the unit interval was resolved into two subsets of upper measure 1, so either of these can be resolved into turn into two subsets of upper measure 1, and so on *ad infinitum.*

To effect such a resolution return to the non-measurable set $S$ of §1. In this we put together groups of points having the abscissas $x \pm m/2^n$. For a fixed value of $x$ and of $n$ a series of $2^n$ equidistant points is obtained. These we will now separate into two groups of $2^{n-1}$ points in the following manner. If, for convenience, the points are denoted in the order of increasing abscissa by $x_1, x_2, \cdots, x_{2^n}$, let the first two points be assigned one to one group and one to the other; let the next two be assigned in the opposite way so that $x_1$ and $x_3$ do

* YOUNG's *Theory of Point Sets*, p. 96. His definition is reconciled with LEBESGUE's definition on p. 106.

† See, in particular, the latter part of YOUNG's Chap. 5.

not belong to the same group, nor $x_2$ and $x_4$; the next four points in a manner opposite to that of assigning the first four so that $x_{i+4}$ and $x_i$ $(i = 1, 2, 3, 4)$ belong to different groups; the next eight so that $x_{i+8}$ and $x_i$ $(i = 1, 2, \cdots, 8)$ do not fall in the same group; and so on. Now $x$ itself, being one of the $2^n$ points $x_i$, falls in one or the other of our two groups. According as the group does or does not contain $x$, we will assign it to one or the other of two components $S'$, $S''$ into which $S$ is to be split. It is seen very easily that this separation of the points $\{x \pm m/2^n\}$ into two groups for fixed $x$ and for two consecutive values of $n$ is consistent the one with the other; in fact, the assignment of the points $\{x \pm m'/2^{n+1}\}$ is completely determined by the previous assignment of the points of its subset $\{x \pm m/2^n\}$.

Let this separation be effected first for any fixed $x$ belonging to $S$ and for all values of $n$. Finally, by varying $x$ within $S$ we effect a decomposition of $S$ into two components $S'$, $S''$. These two components must be of equal upper measure, for by construction $x$ and $x \pm \frac{1}{2}$ belong the one to $S'$ and the other to $S''$. Thus the upper measure of the subset of $S'$ or of $S''$ in the interval $(0, \frac{1}{2})$ is the same as the upper measure of $S''$ or of $S'$ respectively in $(\frac{1}{2}, 1)$.

To secure a quasi-homogeneous structure for $S'$ and $S''$ we will add the requirement that the entire sequence of points $x$, $x/2$, $x/4$, $x/8$, $\cdots$ shall belong to $S'$ (so also to $S''$) whenever any one point $x/2^n$ of the sequence so belongs. This insures that the subsets of $S'$, $S''$ in $(0, 1/2^n)$ shall be similar in distribution to the whole sets $S'$, $S''$ in the unit interval. Furthermore, from our mode of separating the points $(x \pm m/2^n)$ for fixed $n$ and $x$ into two groups, it follows that if the unit interval is divided into $2^n$ equal parts, $2^{n-1}$ of these parts will be identical in respect to distribution of the subsets of $S'$ and $S''$ which they contain and will be similar therefore to the unit interval, as well as the first half of the same. The other $2^{n-1}$ parts are also identical and are obtained from the preceding parts by the interchange of the points of $S'$ and those of $S''$, being similar to the second half of the unit interval. Consider now any portion $ab$ of the unit interval. Two points $a' = m_1/2^n$, $b' = m_2/2^n$ in the interior may be selected as near to $a$ and $b$ as we please, and the intervening intervals consist of $2(m_1 - m_2)$ equal parts of length $1/2^{n+1}$. The half of this number are similar to the first half of unit interval, and the other $m_1 - m_2$ parts are similar to the second half of the unit interval. The argument of § 1 may therefore be applied with immaterial change to the complementary sets $S'$ and $S'' + C + K$. Because $S'$ and $S''$ have by construction equal upper measures at least as great as $\frac{1}{2}$, it follows from § 1 that $S'$ and hence $S''$ is not measurable, and that their common upper measure is 1.

Slight reflection will make it apparent that $S'$, $S''$ can be resolved on similar principles each into two components of upper measure 1, and so on indefinitely.

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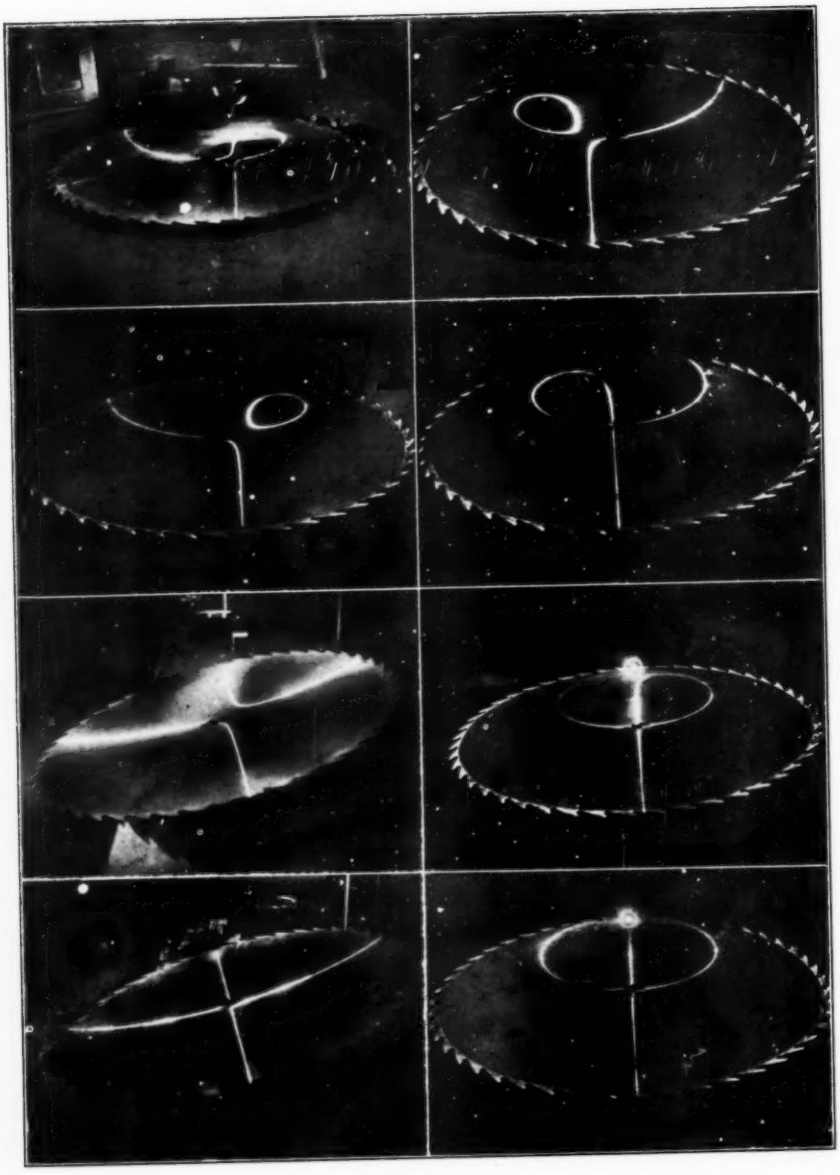

FIG. 1.

[W. H. Roever: Brilliant Points.]